Discrete Orthogonal Structures

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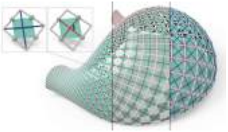
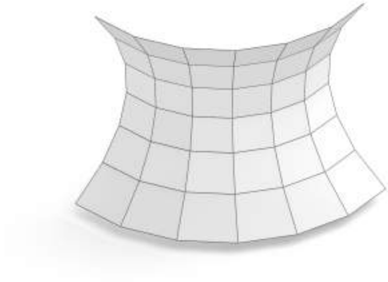
International Geometry Summit, 2023



Motivation



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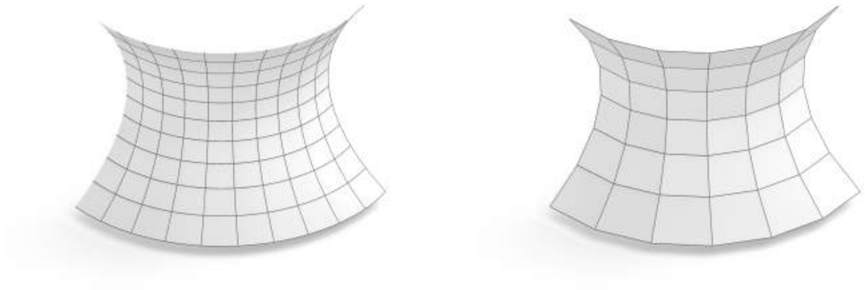
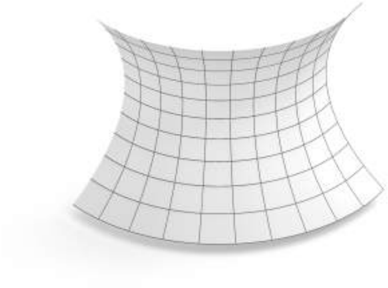
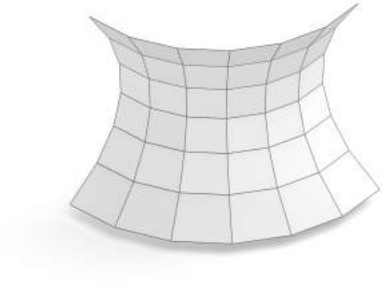


Motivation





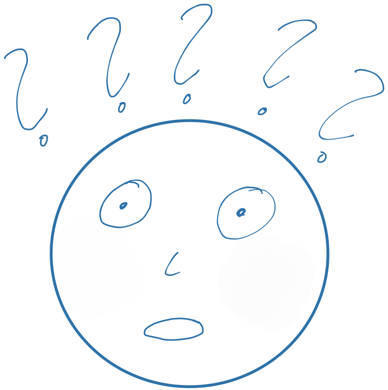
What we want to do:

Find a definition of discrete orthogonality.

Overview

Road to discrete orthogonality

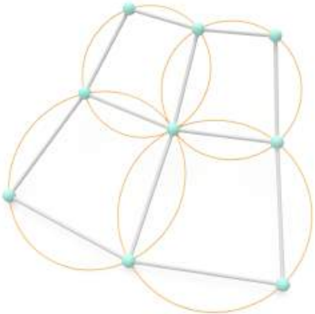
- Some theoretical background
- Orthogonal multi-nets
- Applications



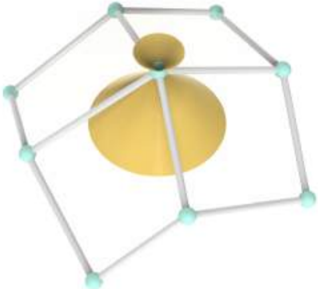
Question

How to define discrete orthogonality?

How to define discrete orthogonality? - Famous examples

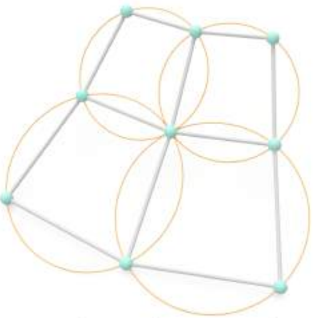


Discrete orthogonality via a circular mesh.

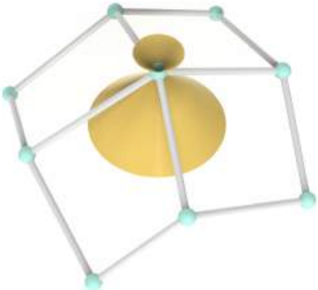


Discrete orthogonality via a conical mesh.

How to define discrete orthogonality? - Famous examples



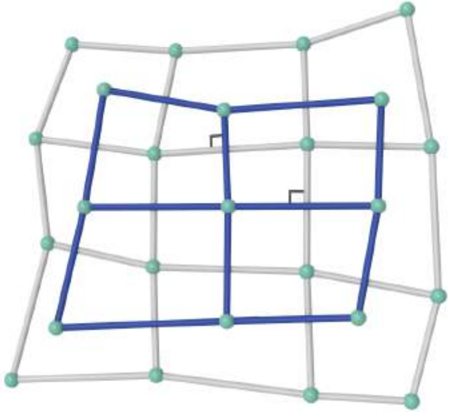
Discrete orthogonality via a circular mesh.



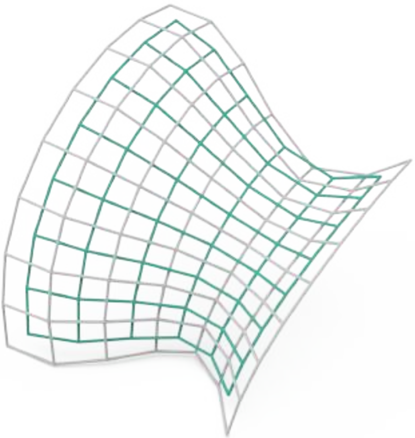
Discrete orthogonality via a conical mesh.

We base our definition on the approach of mesh pairings. [Bobenko et al 2018]

Mesh Pairings

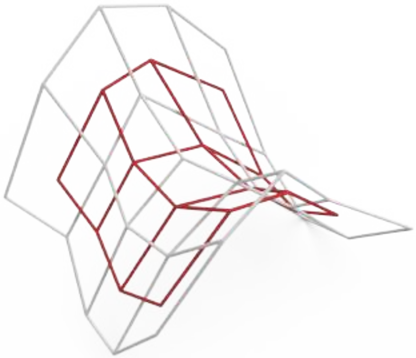


Mesh Pairings



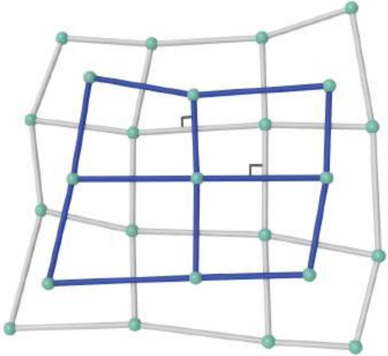
Circular and conical meshes always form principal mesh pairings.
[Pottmann, Wallner 2008]

Mesh Pairings



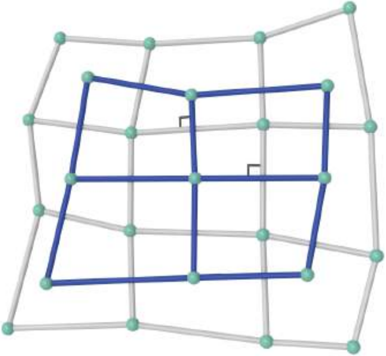
Koenigs meshes in the sense of [Bobenko, Suris 2009] and Koenigs meshes in the sense of [Doliwa 2003] form Koenigs mesh pairings.

Mesh Pairings



Properties

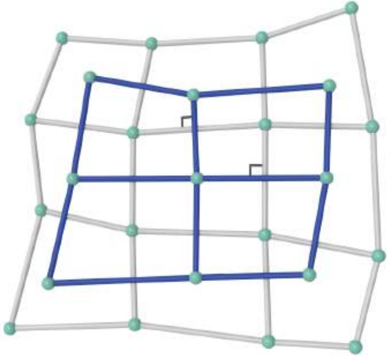
Mesh Pairings



Properties

- Strong theoretical potential

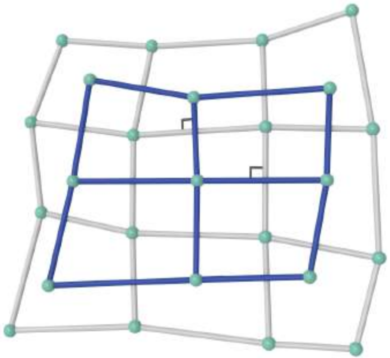
Mesh Pairings



Properties

- Strong theoretical potential
- Offers simple characterizations

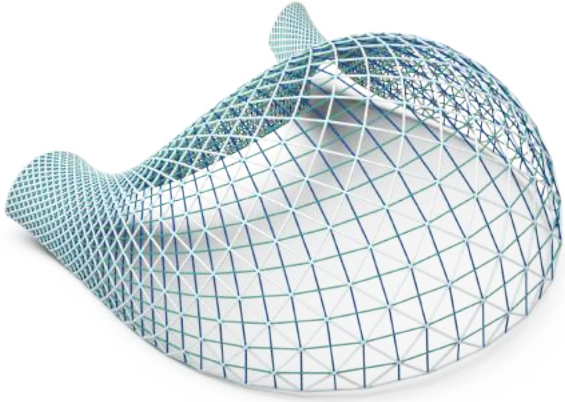
Mesh Pairings



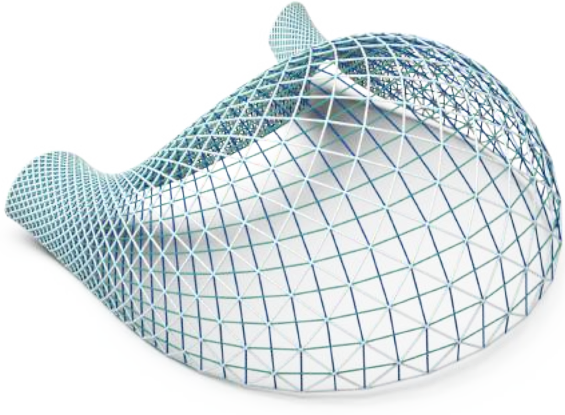
Properties

- Strong theoretical potential
- Offers simple characterizations
- Slightly too many meshes. . .

Mesh Pairings arise as diagonal meshes



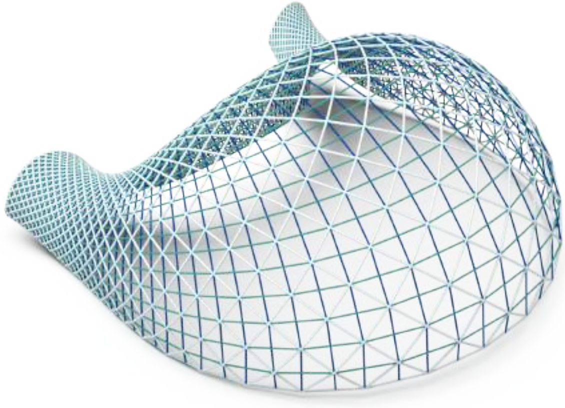
Mesh Pairings arise as diagonal meshes



Lemma

A parametrization is orthogonal if and only if its diagonal parametrization is rhombic.

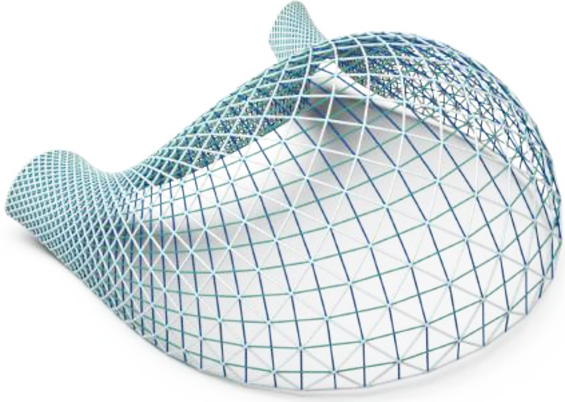
Discrete Orthogonality



Definition

A quadrilateral mesh is orthogonal if its diagonal meshes form a rhombic mesh pairing.

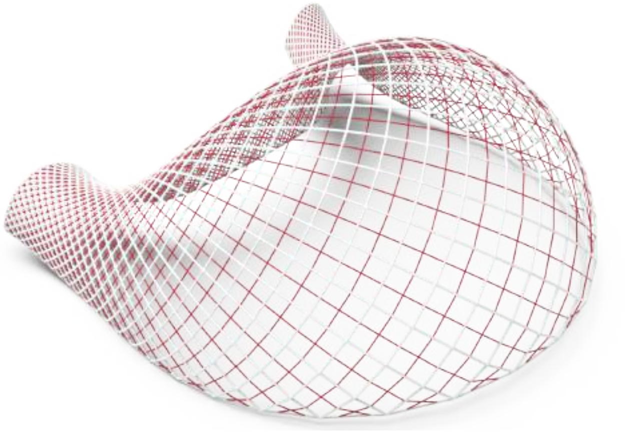
Discrete Orthogonality



Definition

A quadrilateral mesh is orthogonal if the two diagonals in every quad have equal length. - [Wang, Pottmann 2022]

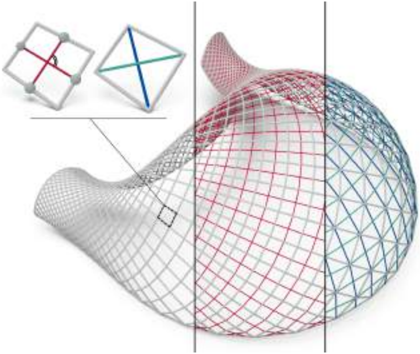
Discrete Orthogonality



Definition

A quadrilateral mesh is orthogonal if the medial lines in every quadrilateral are orthogonal.

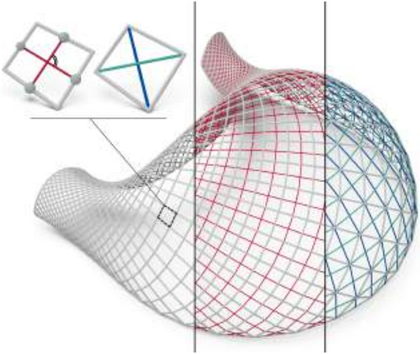
Discrete Orthogonality



Discrete orthogonality

- Defined via equal diagonal lengths

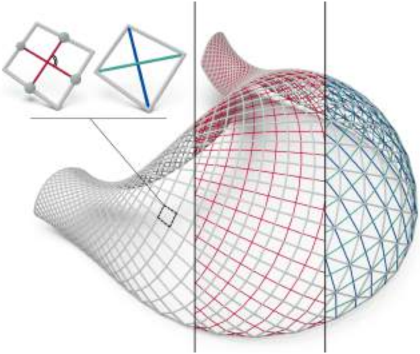
Discrete Orthogonality



Discrete orthogonality

- Defined via equal diagonal lengths
- Observable in the medial lines

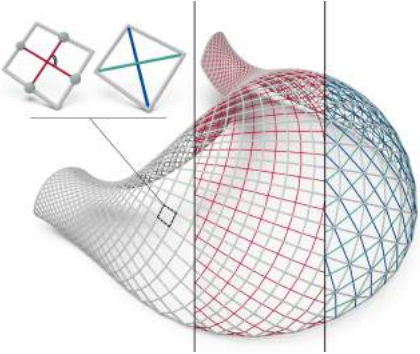
Discrete Orthogonality



Discrete orthogonality

- Defined via equal diagonal lengths
- Observable in the medial lines
- Second order approximation

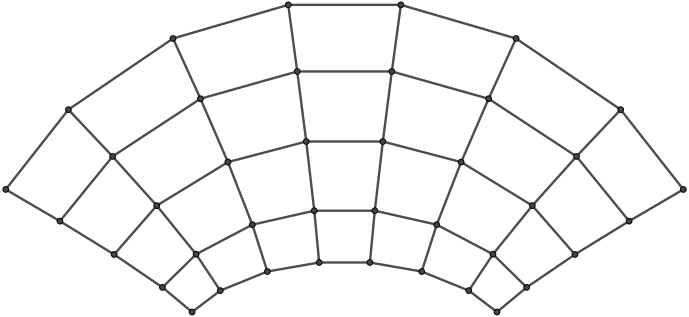
Discrete Orthogonality



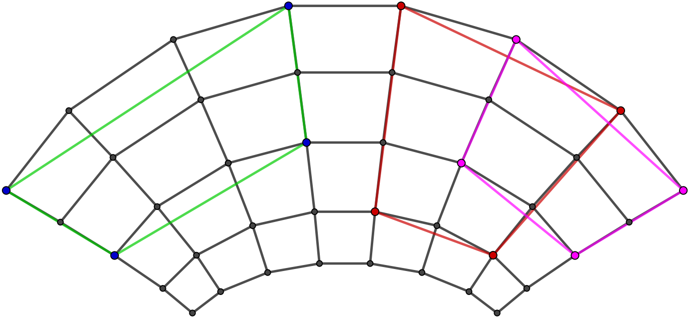
Discrete orthogonality

- Defined via equal diagonal lengths
- Observable in the medial lines
- Second order approximation
- Possible for general quadrilaterals

Orthogonal Multi-Nets



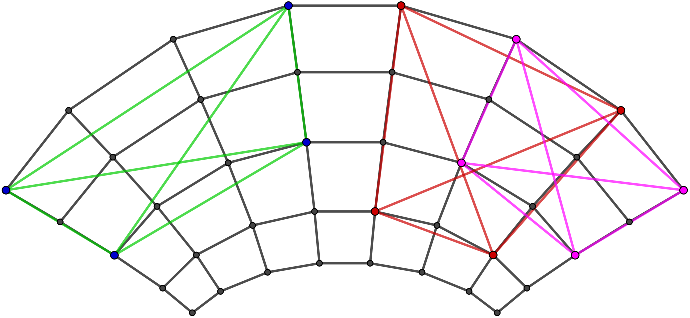
Orthogonal Multi-Nets



Definition

In an *orthogonal multi-net* every combinatorial rectangle is orthogonal.

Orthogonal Multi-Nets



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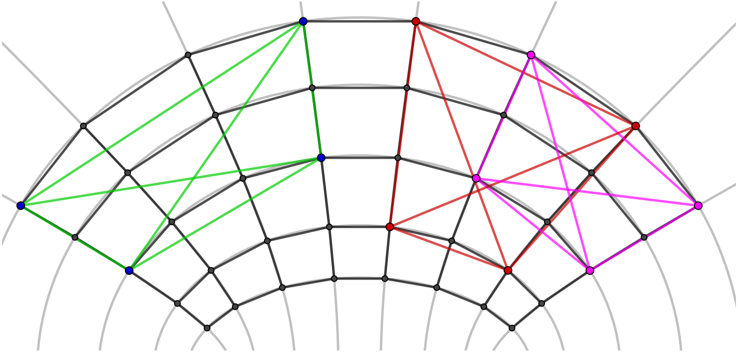
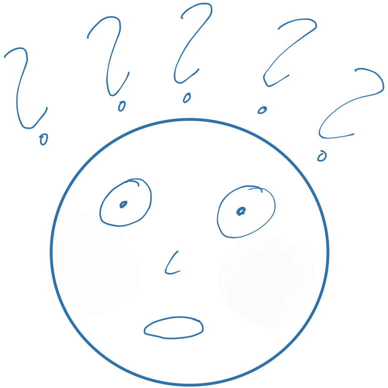
Orthogonal Multi-Nets



Ivory's Theorem

Arcs of confocal conic sections form quadrilaterals with equal diagonal lengths.

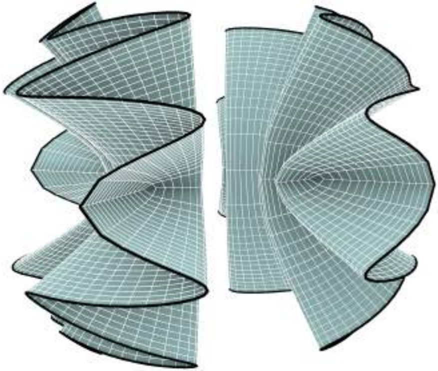
Orthogonal Multi-Nets in Space



Question

Do non-planar orthogonal multi-nets exist as well?

Orthogonal Multi-Nets in Space

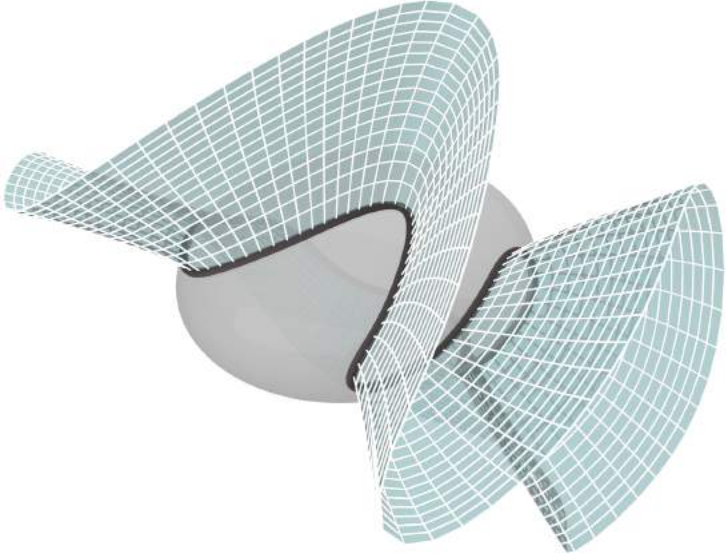


Properties

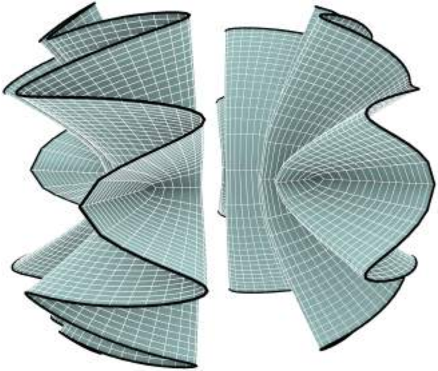
- Polylines lie on confocal quadrics.
- Two Polylines are related by affine mappings.
- The affine mapping is determined by the underlying quadric: $\boldsymbol{x} \mapsto \sqrt{tS + I}\boldsymbol{x}$.

Orthogonal Multi-Nets: Interactive Design

Orthogonal Multi-Nets: Interactive Design



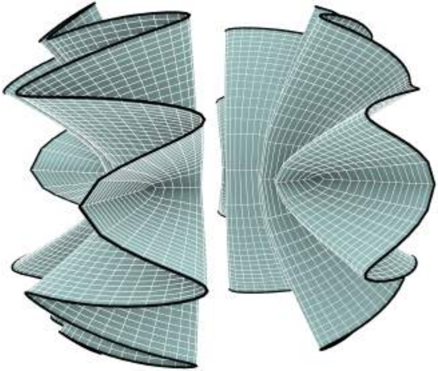
Orthogonal Multi-Nets as Regularizer



Observation

Polylines of one family follow curves of the form $t \mapsto \sqrt{tS + I}\boldsymbol{x}$.

Orthogonal Multi-Nets as Regularizer



Observation

Polylines of one family follow curves of the form $t \mapsto \sqrt{tS + I}\boldsymbol{x}$.

Conjecture

Optimizing an orthogonal mesh towards multi-orthogonality increases the fairness of polylines.

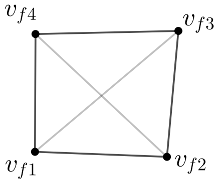
Applications

Energy terms

Applications

Energy terms

$$E_{Ortho} = \sum_{f=1}^{|F|} (\|v_{f1} - v_{f3}\|^2 - \|v_{f2} - v_{f4}\|^2)^2$$

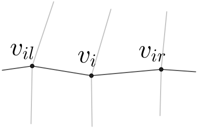


Applications

Energy terms

$$E_{Ortho} = \sum_{f=1}^{|F|} (\|v_{f1} - v_{f3}\|^2 - \|v_{f2} - v_{f4}\|^2)^2$$

$$E_{Fair} = \sum_{i \in polyline} (2v_i - v_{il} - v_{ir})^2$$



Applications

Energy terms

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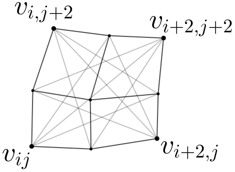
$$E_{Fair} = \sum_{i \in polyline} (2v_i - v_{il} - v_{ir})^2$$

$$E_{Multi} = \sum_{i,j<k,l \leq i,j+2} (\|v_{ij} - v_{kl}\|^2 - \|v_{il} - v_{kj}\|^2)^2$$



Applications

Energy terms

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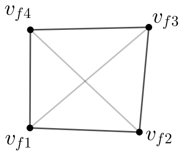
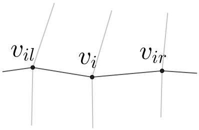
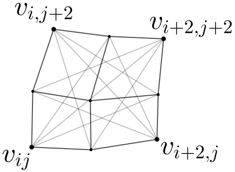
$$E_{Multi} = \sum_{i,j<k,l\leq i,j+2} (\|v_{ij} - v_{kl}\|^2 - \|v_{il} - v_{kj}\|^2)^2$$

Total Energy

$$E = E_{Ortho.} + \omega_1 E_{fair} + \omega_2 E_{locMulti}$$



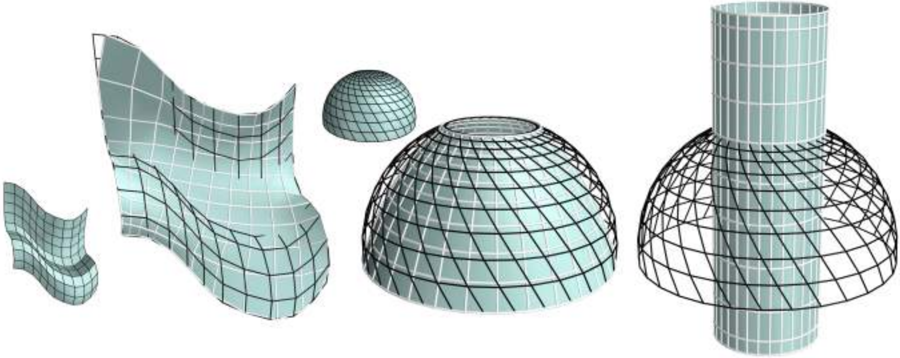




Applications

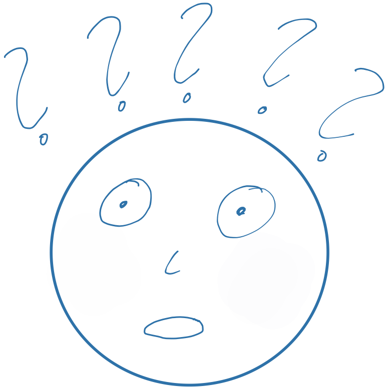
Total Energy

$$E = E_{Ortho.} + \omega_1 E_{fair} + \omega_2 E_{locMulti}$$



(a) (b) (c)

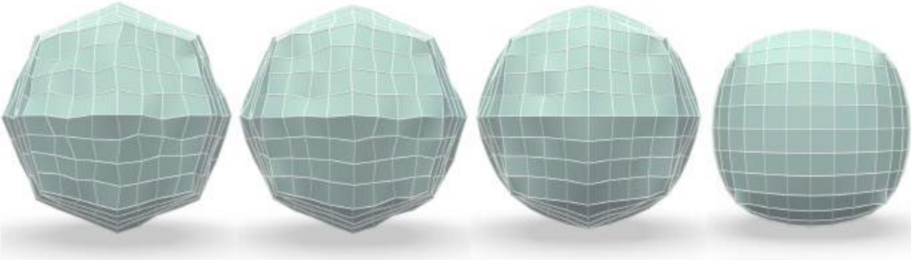
Applications



Question

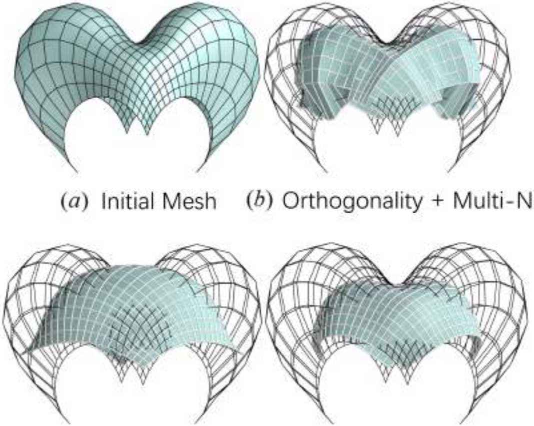
What is the effect of the local multi-net energy?

Orthogonal Multi-Nets as Regularizer



(a) Initial Mesh (b) Orthogonality (c) +Multi-Net (d) +Fairness

Orthogonal multi-nets as regularizer



(a) Initial Mesh (b) Orthogonality + Multi-Net

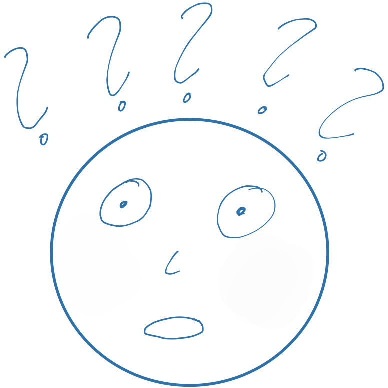
(c) Orthogonality + Fairness (d) Orthogonality + Fairness + Multi-Net

Orthogonal multi-nets as regularizer

Conclusion

The energy term $E_{locMulti}$ is too weak to guide an optimization process towards useful results in general. However, in combination with classical fairness terms it can help to preserve features of the initial meshes.

Applications



Question

What else can we do with discrete orthogonality?

Applications: Principal curvature lines

Total Energy

$E = E_{Ortho.} + \omega_1 E_{fair} + \omega_2 E_{locMulti} + E_{PQ}$

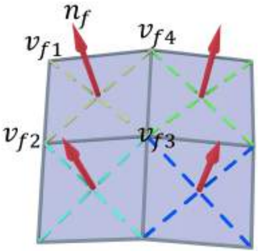
Idea

We add discrete conjugacy expressed through planar quadrilaterals. This yields a discrete version of principal curvature lines. (I.e. the lines of maximal/minimal curvature in a surface.)

Applications: Principal curvature lines

Total Energy

$$E = E_{Ortho.} + \omega_1 E_{fair} + \omega_2 E_{locMulti} + E_{PQ}$$



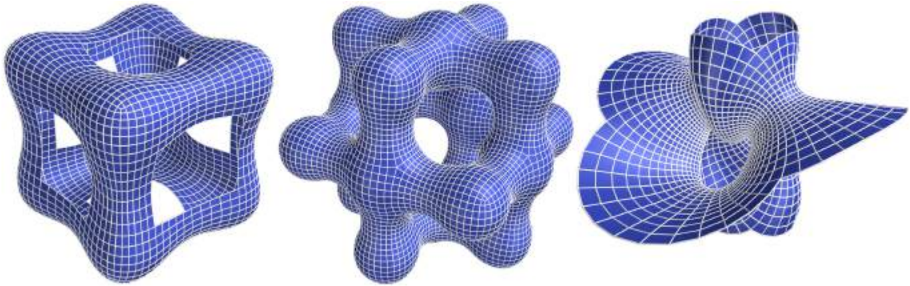
Energy term for planarity

$$E_{PQ} = \sum_{f=1}^{|F|} \sum_{j=1}^{4} (n_f \cdot (v_{fj} - v_{fk}))^2 + \\ + \sum_{f=1}^{|F|} (n_f \cdot n_f - 1)^2$$

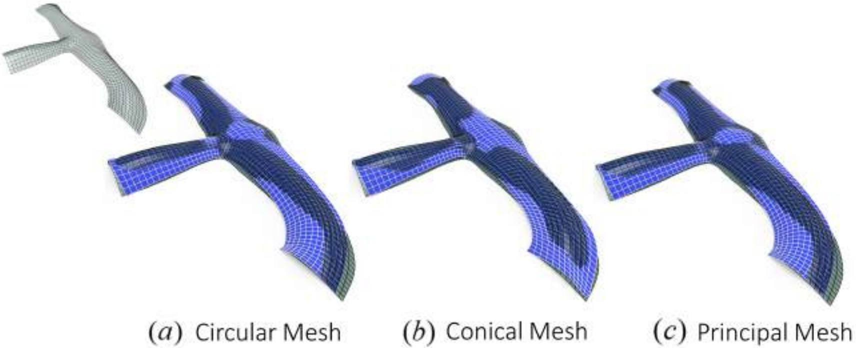
Applications: Principal curvature lines

Total Energy

$$E = E_{Ortho.} + \omega_1 E_{fair} + \omega_2 E_{locMulti} + E_{PQ}$$



Applications: Principal curvature lines



Applications: Developable Surfaces

Total Energy

$$E = E_{Ortho.} + \omega_1 E_{fair} + \omega_2 E_{locMulti} + E_{Gnet}$$

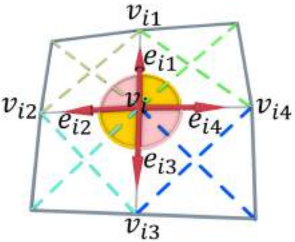
First Idea

Orthogonal geodesics only exist in developable surfaces. Thus, a discrete model of orthogonal geodesics yields a discrete developable surface.

Applications: Developable Surfaces

Total Energy

$$E = E_{Ortho.} + \omega_1 E_{fair} + \omega_2 E_{locMulti} + E_{Gnet}$$



Energy term for geodesics

$$E_{Gnet} = \sum_{i=1}^{|V|} ((e_{i1} \cdot e_{i2} - e_{i3} \cdot e_{i4})^2 + (e_{i2} \cdot e_{i3} - e_{i4} \cdot e_{i1})^2) + \sum_{i=1}^{|V|} \sum_{j=1}^{4} \left(e_{ij} - \frac{v_{ij} - v_i}{\|v_{ij} - v_i\|} \right)^2$$

Applications: Developable Surfaces

Total Energy

$$E = E_{Ortho.} + \omega_1 E_{fair} + \omega_2 E_{locMulti} + E_{Cheby}$$

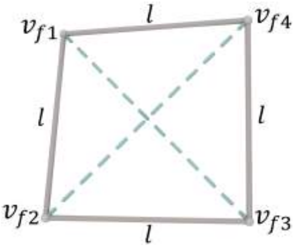
Second Idea

An isogonal Chebyshev net has vanishing Gaussian curvature. Thus, modelling a discrete orthogonal Chebyshev net yields a discrete developable surface.

Applications: Developable Surfaces

Total Energy

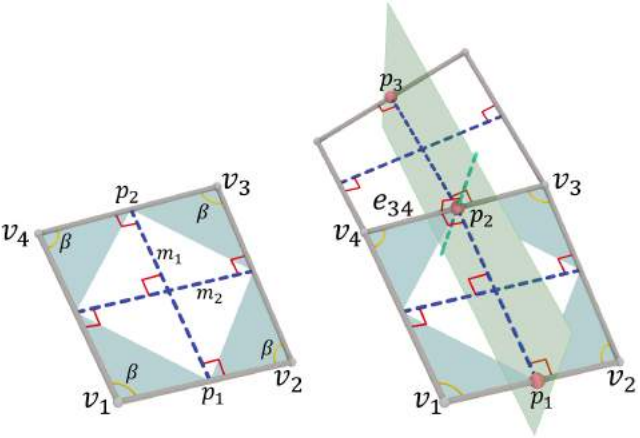
$$E = E_{Ortho.} + \omega_1 E_{fair} + \omega_2 E_{locMulti} + E_{Cheby}$$



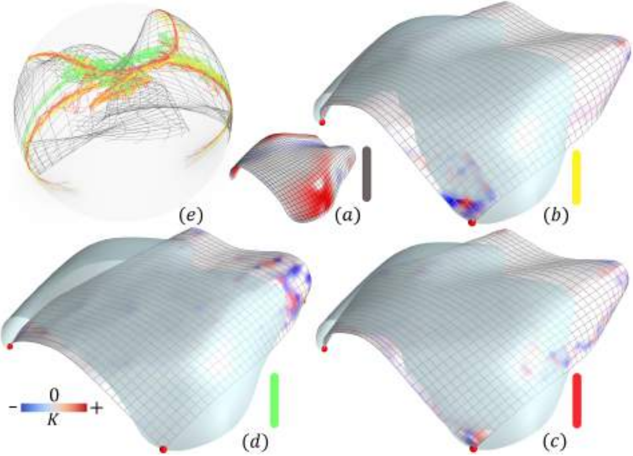
Energy term for Chebyshev nets

$$E_{Cheby} = \sum_{e_{ij}}^{|E|} ((v_i - v_j)^2 - l^2)^2$$

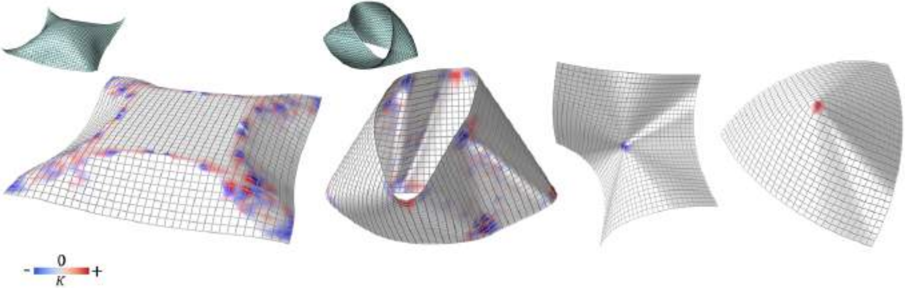
Applications: Developable Surfaces



Applications: Developable Surfaces



Applications: Developable Surfaces



Applications: Minimal Surfaces

Total Energy

$E = E_{Ortho.} + \omega_1 E_{fair} + \omega_2 E_{locMulti} + E_{Anet}$

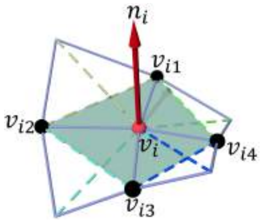
Idea

An orthogonal asymptotic net (A-net) has zero mean curvature. Thus, a discrete orthogonal asymptotic mesh yields a discrete minimal surface.

Applications: Minimal Surfaces

Total Energy

$$E = E_{Ortho.} + \omega_1 E_{fair} + \omega_2 E_{locMulti} + E_{Anet}$$



Energy term for A-nets

$$E_{Anet} = \sum_{i=1}^{|V|} \sum_{j=1}^{4} (n_i \cdot (v_{ij} - v_i))^2 + \sum_{i=1}^{|V|} (n_i \cdot n_i - 1)^2$$

Applications: Minimal Surfaces

Total Energy

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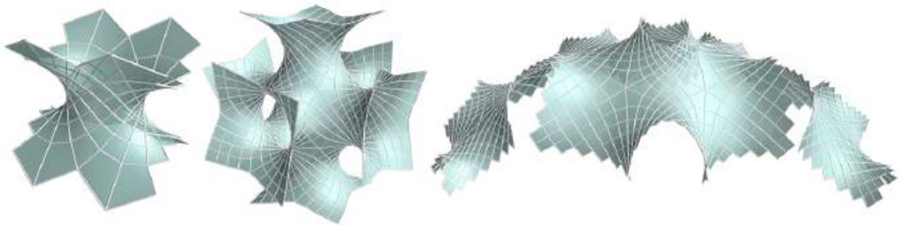
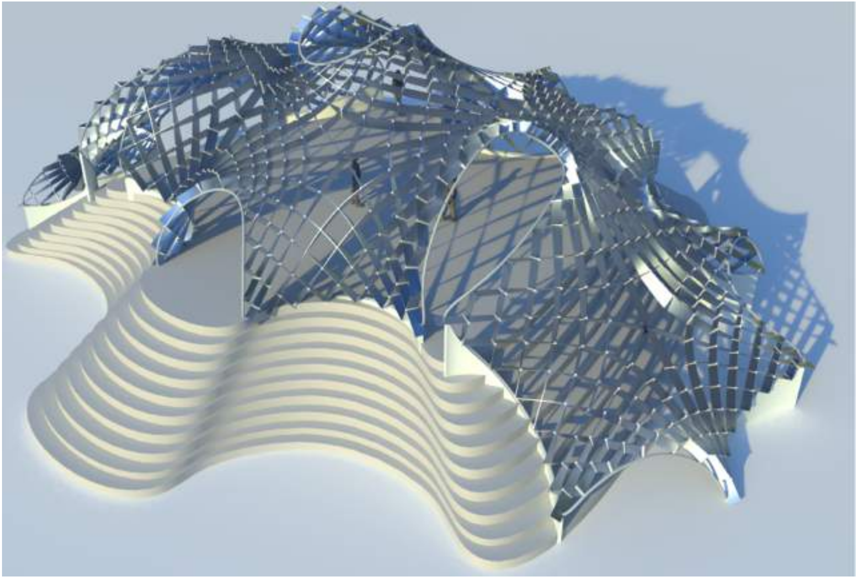


Applications: Minimal Surfaces



Applications: CMC Surfaces

Total Energy

$$E = E_{Ortho.} + \omega_1 E_{fair} + \omega_2 E_{locMulti} + E_{Snet}$$

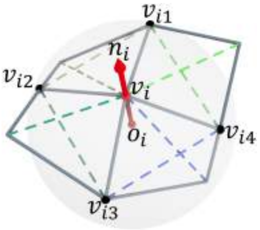
Idea

Curves that bisect the principal directions osculate the same sphere (*Meusnier sphere*) determined by their their normal curvature in the point of intersection. If the curves are also orthogonal their normal curvature coincides with the mean curvature. Thus, if the corresponding *Meusnier spheres* have constant radius the mean curvature is constant as well.

Applications: CMC Surfaces

Total Energy

$$E = E_{Ortho.} + \omega_1 E_{fair} + \omega_2 E_{locMulti} + E_{Snet}$$



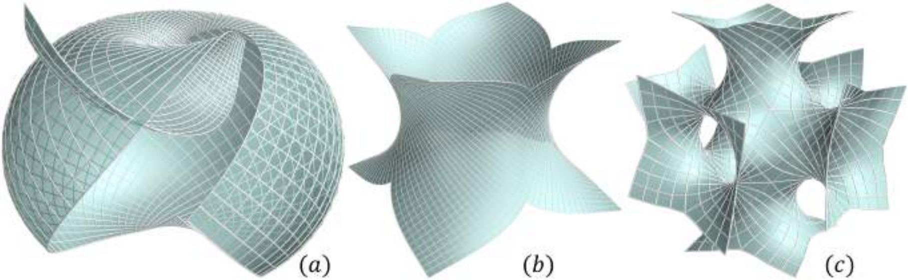
Energy term for S-nets

$$E_{Snet} = \sum_{i=1}^{|V|} \sum_{j=1}^{4} ((v_{ij} - o_i)^2 - R^2)^2 + \sum_{i=1}^{|V|} ((v_i - o_i)^2 - R^2)^2$$

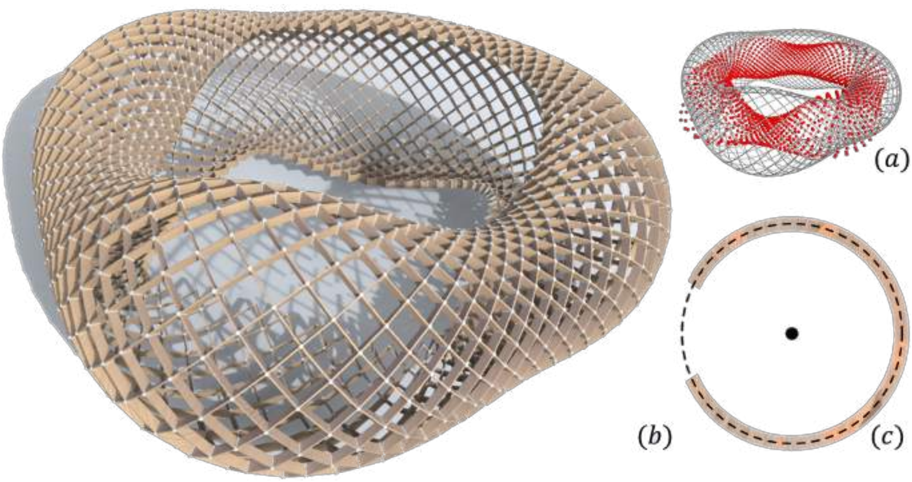
Applications: CMC Surfaces

Total Energy

$$E = E_{Ortho.} + \omega_1 E_{fair} + \omega_2 E_{locMulti} + E_{Snet}$$



Applications: CMC-Surfaces



Applications: Principal Stress Nets

Total Energy

$E = E_{Ortho.} + \omega_1 E_{fair} + \omega_2 E_{locMulti} + E_{Equi}$

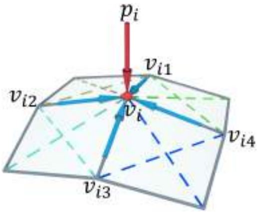
Idea

Orthogonal meshes in equilibrium constitute discrete versions of principal stress nets.

Applications: Principal Stress Nets

Total Energy

$$E = E_{Ortho.} + \omega_1 E_{fair} + \omega_2 E_{locMulti} + E_{Equi}$$

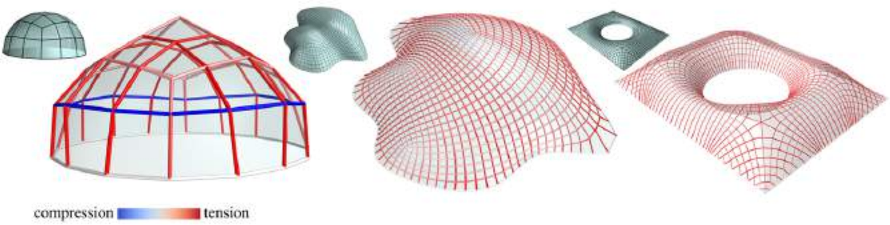


Energy term for equilibrium

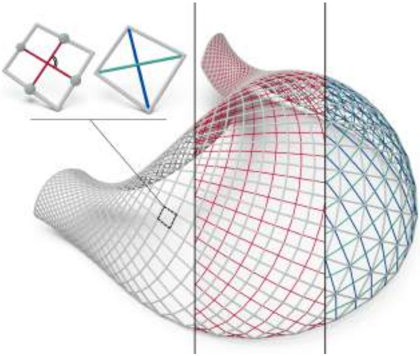
$$c_{load,i} = \sum_{j=1}^{4} w_{ij}(v_i - v_{ij}) - \begin{pmatrix} 0 \\ 0 \\ p_i \end{pmatrix}$$

$$E_{Equi} = \sum_{i=1}^{|V|} c_{load,i}^2$$

Applications: Principal Stress Nets



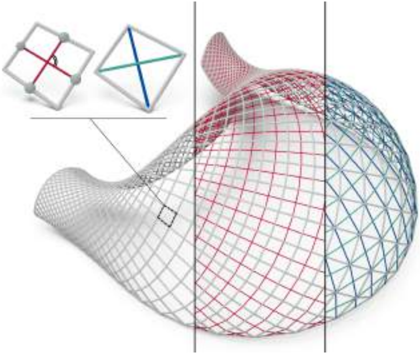
Summing up



Discrete Orthogonality

- Defined via equal diagonal lengths.
- Definition is based on rhombic mesh pairings.
- Related to orthogonal Multi-nets via *Ivory's theorem*.
- Easily incorporated into numerical optimization.
- Applicable to non-planar quad meshes allowing a wide range of applications.

Summing up



Thank you for your attention!

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