Discrete Isogonal Nets with Similar Parallelograms

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Abstract

High-quality surface designs are increasingly significant in industrial applications, such as architecture and product design, yet they pose challenges in balancing visual appeal and functional requirements.

Isogonal nets (I-nets) stand out for their aesthetically pleasing patterns and engineering practicality. However, constructing such nets remains difficult due to their dependence on complex angle constraints or a narrow focus on orthogonal scenarios.

We propose a novel representation and construction method for I-nets characterized by similar mid-edge subdivided parallelograms in the quad faces. This approach achieves a simple yet versatile representation that generalizes orthogonal nets and extends to the construction of **isogonal 4-webs (I-webs)**. By focusing on constraining edge ratios, our method enables efficient integration into mesh optimization algorithms.

We demonstrate the effectiveness of I-nets and I-webs in freeform shapes through conformal mapping and numerical optimization. Experiments on various surfaces validate our method, showcasing its potential for both theoretical advancements and practical applications.

Parallelogram determined by two edge ratios

Given a parallelogram $ABCD$ with neighboring edge lengths $BC = a$ and $AB = b$, their interior angle $\angle B = \theta$ and diagonal angle $\angle AOD = \theta_0$, the lengths of the two diagonal $AC = p$ and $BD = q$ can be expressed using the Law of Cosine in trigonometry as follows:

$$p^2 = a^2 + b^2 - 2ab\cos\theta, \quad q^2 = a^2 + b^2 + 2ab\cos\theta,$$

which brings to

$$p^2 + q^2 = 2(a^2 + b^2).$$

A parallelogram is called an *(θ, θ_0)-Parallelogram*, if we name the interior angle as θ and diagonal angle as θ_0.

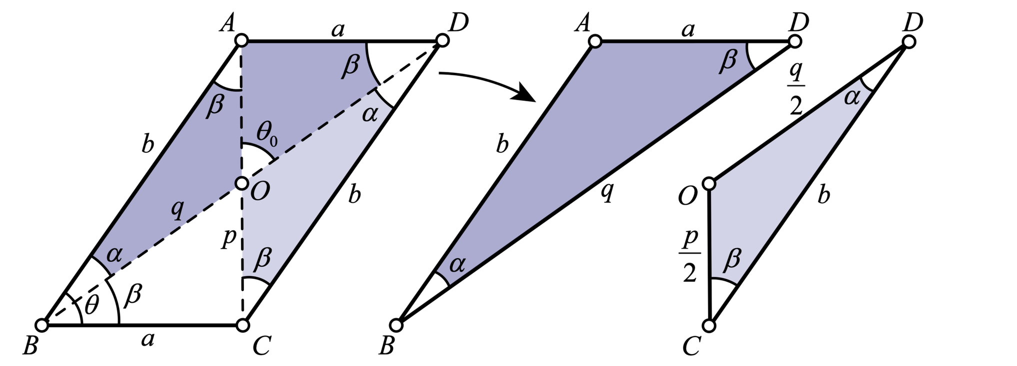

Figure 1: A (θ, θ_0)-Parallelogram with an interior angle θ and an diagonal angle θ_0. If $\theta = \theta_0$, it is a θ-Parallelogram, which possesses two similar triangles $\triangle ABD \simeq \triangle ODC$ with equal edge ratios $a : b : q = \frac{p}{2} : \frac{q}{2} : b$.

For a (θ, θ_0)-Parallelogram with two edge lengths $a = 1$ and $b = \lambda$ and two diagonal lengths p and q, the interior angles θ and the diagonal angle θ_0 can be expressed in terms of λ and μ as follows:

$$\cos\theta = \frac{(1+\lambda^2)(\mu^2-1)}{2\lambda(1+\mu^2)}, \quad \cos\theta_0 = \frac{(1+\mu^2)(\lambda^2-1)}{2\mu(1+\lambda^2)},$$

where λ and μ satisfy the conditions:

$$\frac{|\mu-1|}{\mu+1} < \lambda < \frac{\mu+1}{|\mu-1|}, \quad \frac{|\lambda-1|}{\lambda+1} < \mu < \frac{\lambda+1}{|\lambda-1|}.$$

Additionally, the angles α and β of the triangle forming half of the parallelogram can be computed by λ and μ as follows:

$$\cos\alpha = \frac{3\lambda^2\mu^2 + \lambda^2 + \mu^2 - 1}{2\lambda\mu\sqrt{2(1+\lambda^2)(1+\mu^2)}}, \quad \cos\beta = \frac{\lambda^2\mu^2 - \lambda^2 + 3\mu^2 + 1}{2\mu\sqrt{2(1+\lambda^2)(1+\mu^2)}}.$$

Parallelogram with $\theta = \theta_0$

A parallelogram is called a *θ-Parallelogram* if the interior angle θ is equal to the angle between the two diagonals.

For a θ-Parallelogram with two edge lengths $a = 1$ and $b = \lambda$ and two diagonal lengths p and q, the following properties are exhibited:

$$p = \sqrt{2}, q = \sqrt{2}\lambda,$$
$$\cos\theta = \frac{1}{2}(\lambda - \frac{1}{\lambda}),$$
$$\sqrt{2} - 1 < \lambda < \sqrt{2} + 1,$$
$$\cos\alpha = \frac{\sqrt{2}(3\lambda^2 - 1)}{4\lambda^2}, \cos\beta = \frac{\sqrt{2}(1+\lambda^2)}{4\lambda}.$$

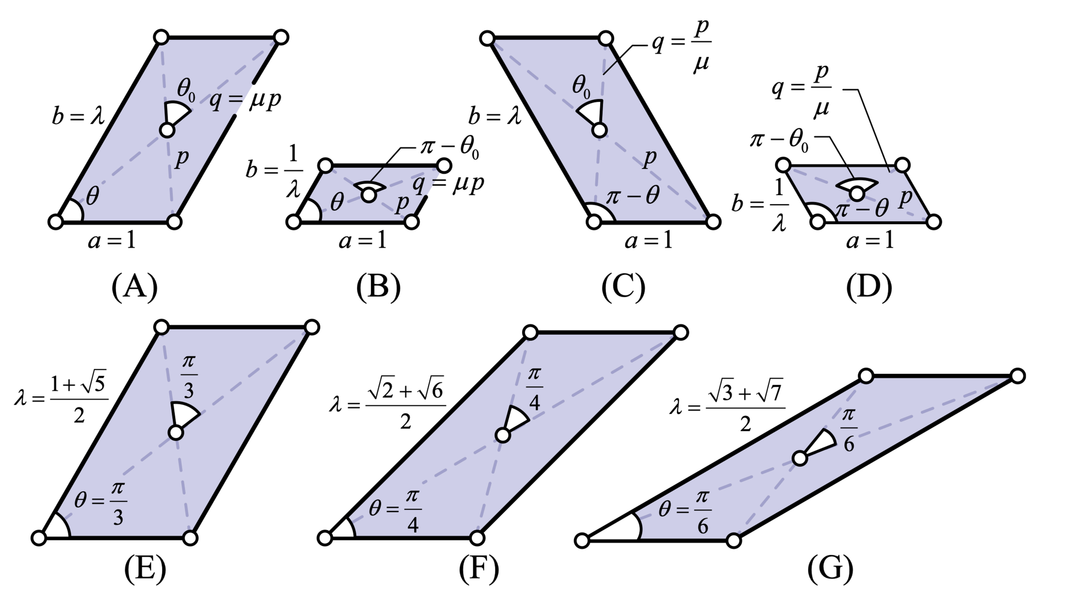

Figure 2: $(A-D)$ (θ, θ_0)-Parallelograms with different edge ratios $(b : a, q : p)$: (A) (λ, μ), (B) $(\frac{1}{\lambda}, \mu)$, (C) $(\lambda, \frac{1}{\mu})$, (D) $(\frac{1}{\lambda}, \frac{1}{\mu})$. $(E-G)$ θ-Parallelograms with $\theta = \frac{\pi}{3}, \frac{\pi}{4}$, and $\frac{\pi}{6}$, respectively.

Reference

[1] Felix Dellinger, Xinye Li, and Hui Wang#.
Discrete orthogonal structures.
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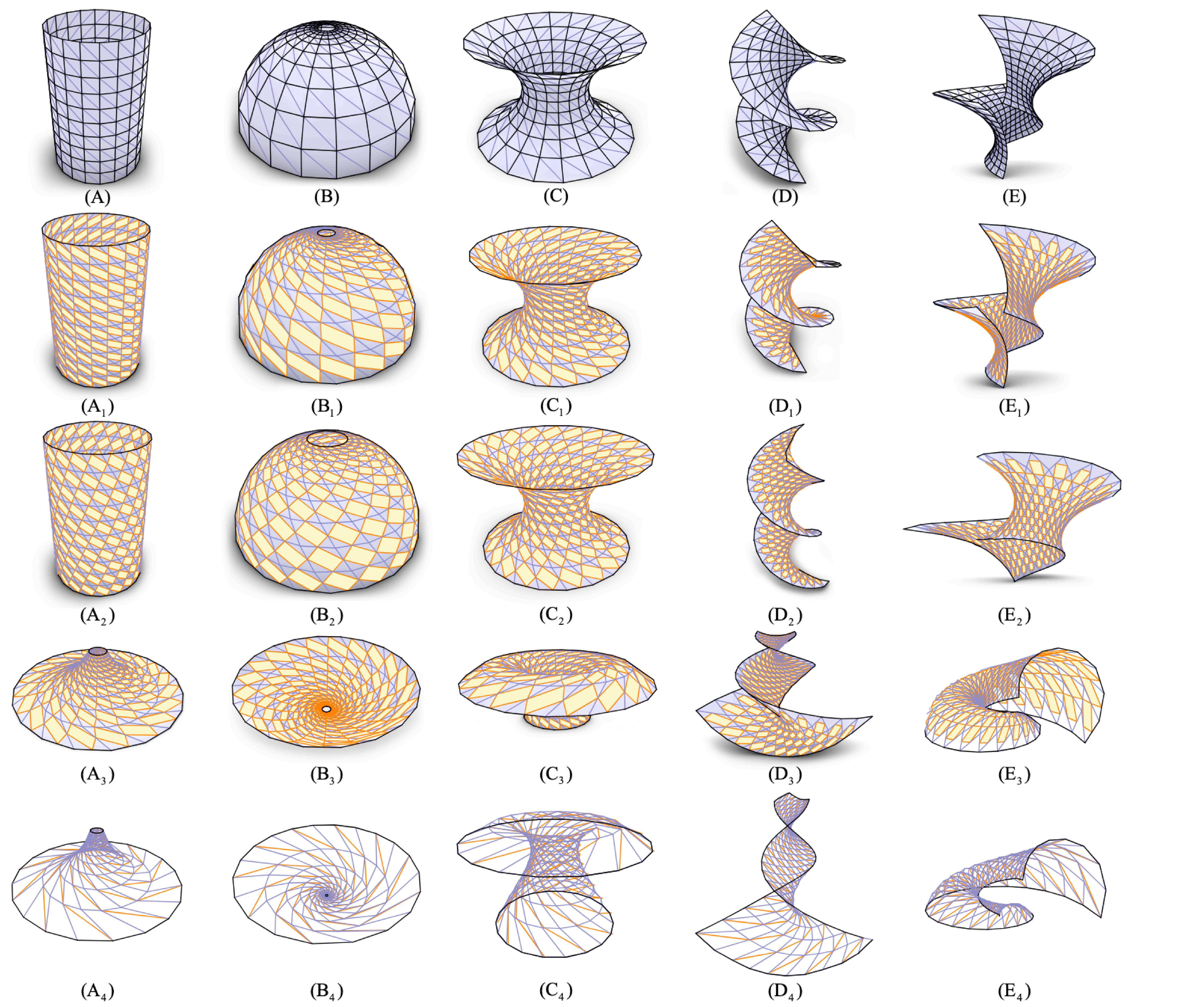

Figure 3: $(A_1 - E_1)$ I-nets with similar inscribed parallelograms, characterized by angles $(\theta, \theta_0) = (\arccos\frac{1}{\sqrt{5}}, \frac{\pi}{4})$ can be extracted from the isothermal parametrization on surfaces. Examples include (A) a cylinder, (B) a sphere, (C) a catenoid, (D) a helicoid, and (E) a conjugate minimal surface combined by catenoid and helicoid. These I-nets can be further optimized to have $(\theta, \theta_0) = (\frac{4\pi}{9}, \frac{\pi}{4})$ in $(A_2 - E_2)$, approximating the corresponding surfaces. Alternatively, they can be transformed conformally under Möbius transformations to new surface shapes in $(A_3 - E_3)$, and then optimized to have $\theta = \theta_0 = \frac{5\pi}{18}$ approximated on the surfaces in $(A_4 - E_4)$.

I-net and I-web defined based on mid-edge subdivided parallelograms

In a general spatial quad, connecting the midpoints of the edges in a counterclockwise direction forms a nested subdivided parallelogram [1]. This parallelogram has opposite edges parallel and equal in length to $\mathbf{d}_1$ and $\mathbf{d}_2$. Thus, θ equals an interior angle of the parallelogram, and θ_0 is the angle between its diagonals.

A quad is called a $(\theta(\lambda,\mu), \theta_0(\lambda,\mu))$-*quad*, short as *$(\theta, \theta_0)$-quad*, if its mid-edge subdivided parallelogram is a (θ, θ_0)-Parallelogram.

A quad is called a $\theta(\lambda)$-*quad*, short as *θ-quad*, if its mid-edge subdivision is a θ-Parallelogram.

Proposition 1. For a (θ, θ_0)-quad with two diagonal vectors $\mathbf{d}_1$ and $\mathbf{d}_2$ and two medial-line vectors $\mathbf{m}_{13}$ and $\mathbf{m}_{24}$, let λ and μ be the ratios of the quad's two diagonal lengths and two medial lines, respectively, given by

$$\frac{\|\mathbf{d}_2\|}{\|\mathbf{d}_1\|} = \lambda, \quad \frac{\|\mathbf{m}_{13}\|}{\|\mathbf{m}_{24}\|} = \mu.$$

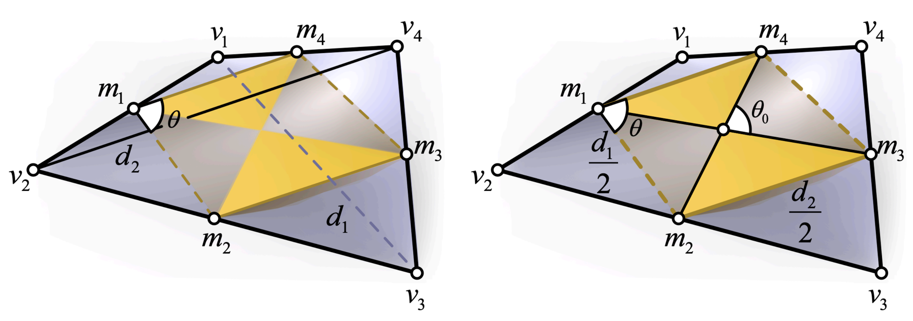

Figure 4: Mid-edge subdivided parallelogram within a general spacial (θ, θ_0)-quad. The interior angle θ and the diagonal angle θ_0 of the parallelogram are functions of the ratio λ of the diagonal lengths and the ratio μ of the medial-line lengths of the quad. The quad is referred to as a θ-quad, if the parallelogram is a θ-Parallelogram, where the angles $\theta = \theta_0$.

Theorem 1. A smooth surface parametrization $\mathbf{X} : (u,v) \subset \mathbb{R}^2 \mapsto \mathbf{X}(u,v) \subset \mathbb{R}^3$ is isogonal (*I-net*) if the ratios $\frac{\|\mathbf{X}_u\|}{\|\mathbf{X}_v\|}$ and $\frac{\|\mathbf{Y}_v\|}{\|\mathbf{Y}_u\|}$ are constant, where $\mathbf{Y}(u,v) = \mathbf{X}(u-v, u+v)$ is the diagonal parametrization of $\mathbf{X}(u,v)$. Furthermore, $\mathbf{Y}(u,v)$ is isogonal with respect to $\mathbf{X}(u,v)$.

Theorem 2. A smooth surface parametrization $\mathbf{X}(u,v)$ and its diagonal parametrization $\mathbf{Y}(u,v) = \mathbf{X}(u-v, u+v)$ form an isogonal 4-web (*I-web*) if the ratios $\frac{\|\mathbf{X}_u\|}{\|\mathbf{X}_v\|}$ and $\frac{\|\mathbf{Y}_v\|}{\|\mathbf{Y}_u\|}$ are constant.

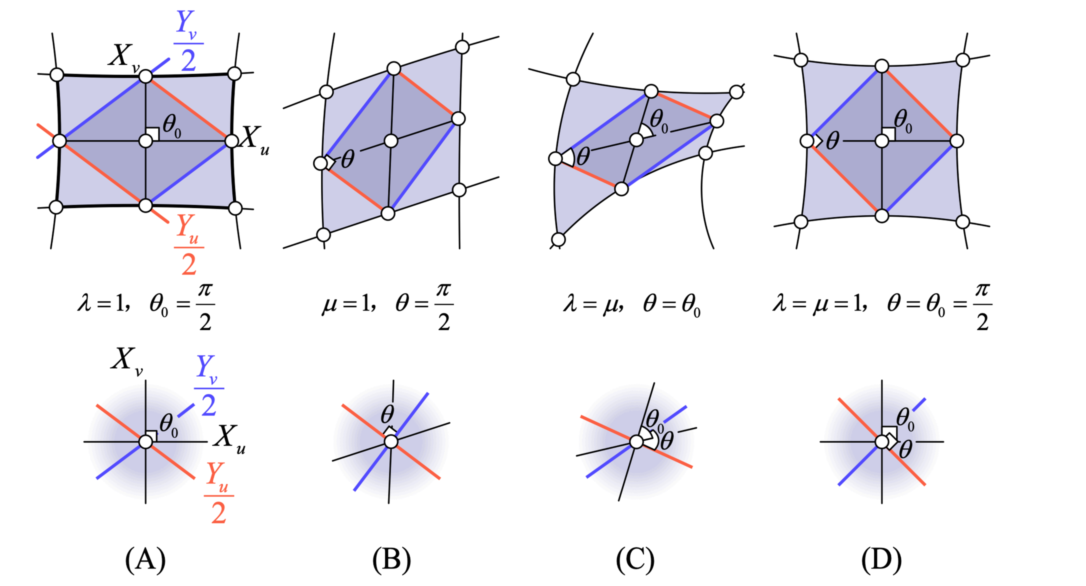

Figure 5: Special cases for I-net and I-web based on the representation of similar parallelograms in the CBP. (A) $\lambda = 1 \Leftrightarrow \theta_0 = \frac{\pi}{2}$. I-net with similar rhombuses becomes O-net. (B) $\mu = 1 \Leftrightarrow \theta = \frac{\pi}{2}$. I-net with similar rectangles. (C) $\lambda = \mu \Leftrightarrow \theta = \theta_0$. I-net with similar θ-Parallelograms. $\mathbf{X}(u,v)$ and $\mathbf{Y}(u,v)$ are aligned through rotation. (D) $\lambda = \mu = 1 \Leftrightarrow \theta = \theta_0 = \frac{\pi}{2}$. O-net with similar squares. $\mathbf{X}(u,v)$ is locally equivalent to $\mathbf{Y}(u,v)$ after a $\frac{\pi}{4}$ rotation.

Optimized I-nets and I-webs on freeform surfaces

The *target function* E_+ for a discrete I-net (*DI-net*) is formulated by integrating local constraints across the mesh and combining them with fairness E_{fair}, self-closeness E_{clos}, approximation E_{clos} and boundary-gliding E_{app} terms:

$$E_+ = E_\square + \omega_1 E_* + \omega_2 E_{fair} + \omega_3 E_{clos} + \omega_4 E_{app} + \omega_5 E_{bdry},$$

where $E_\square$ represents the I-net energy term $\frac{\|\mathbf{d}_2\|}{\|\mathbf{d}_1\|} = \lambda, \frac{\|\mathbf{m}_{13}\|}{\|\mathbf{m}_{24}\|} = \mu$ of each quad face, E_*represents user-defined constraints with ω_1 being 0 or 1, and

$$E_{fair} = \sum_{i \in polyline} (2\mathbf{v}_i - \mathbf{v}_{il} - \mathbf{v}_{ir})^2,$$
$$E_{clos} = \sum_{i \in vertices} (\mathbf{v}_i - \tilde{\mathbf{v}}_i)^2,$$
$$E_{app} = \sum_{i \in vertices} ((\mathbf{v}_i - \mathbf{p}_i) \cdot \mathbf{n}_i)^2,$$
$$E_{bdry} = \sum_{i \in boundary} ((\mathbf{v}_i - \mathbf{p}_i) \times \mathbf{e}_{i1})^2.$$

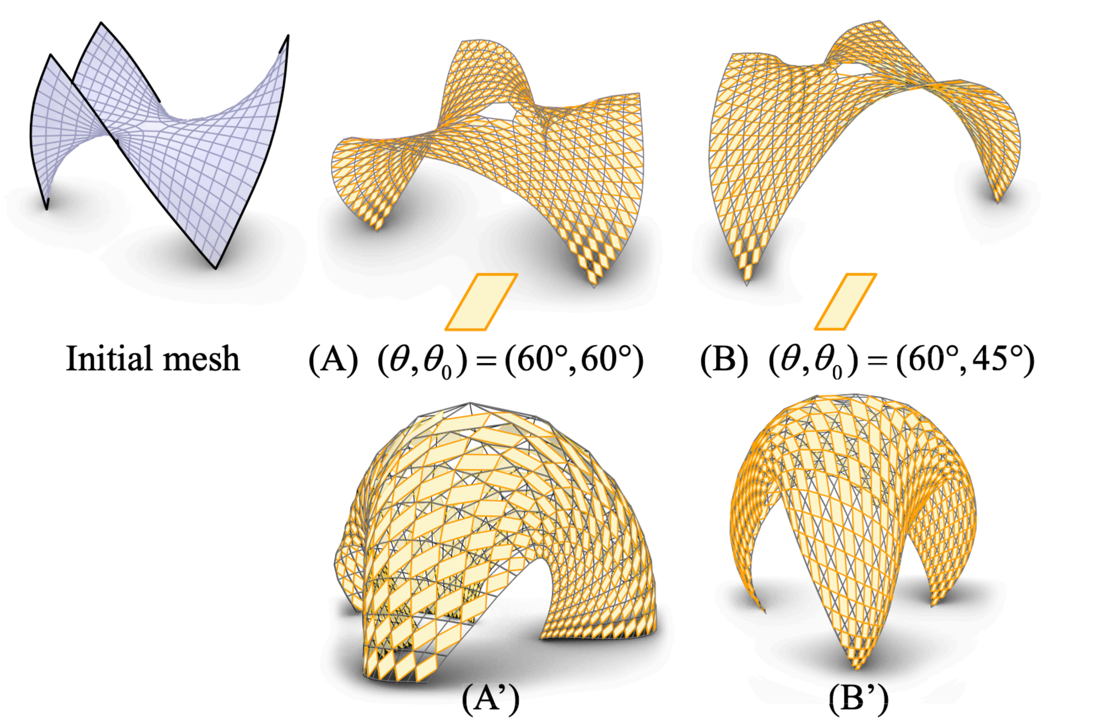

Figure 6: A quad mesh with a singular face at the center is optimized to (A) a DI-net with $\theta = \theta_0 = \frac{\pi}{3}$ and (B) a DI-net with $(\theta, \theta_0) = (\frac{\pi}{3}, \frac{\pi}{4})$, respectively. (A', B') show their corresponding Möbius transformations, preserving the geometric connectivity.