

Quad-mesh based isogonal nets and computational design

ANONYMOUS AUTHOR(S)

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Fig. 1. A freeform surface with planar rectangular boundaries is tiled by an isogonal quad mesh composed of similar parallelograms in a checkerboard pattern. Two similar layouts are obtained under two prescribed control ratios. At the computed ratio $t = 0.412485$, the upper layout features yellow and green parallelograms with identical interior and diagonal angles arranged in mirrored similarity. At $t = 0.5$, the lower layout adopts a half-and-half pattern, where the entire structure is formed by similar rhombuses, half of which are replaced by their area-minimizing circumscribed ellipses. Physical prototypes show good agreement with numerical results. The similarity of laser-cut acrylic panels (parallelograms, rhombuses, and ellipses) is clearly observable in the top view.

We propose a quad-mesh-based framework for discrete isogonal nets using checkerboard patterns with inscribed parallelograms derived from quad meshes. By controlling the angles, similarity, or congruence of these parallelograms, we obtain three classes of isogonal nets on diagonal or control meshes. These nets admit simple explicit representations, enable diverse shape exploration related to conformal mappings with preserved pattern shapes, and support efficient computation and optimization. The third class serves as a discrete model of developable surfaces, grounded in our analysis of smooth isogonal Chebyshev nets and verified through discrete differential geometric quantities. We demonstrate the effectiveness of the three proposed nets through pattern designs for shell structures and interactive paper-cutting models, both fabricated with repetitive similar elements.

CCS Concepts: • **Computing methodologies** → **Shape modeling**; **Optimization algorithms**.

Additional Key Words and Phrases: Isogonal net, Checkerboard pattern, Quad meshes, Developable surface, Conformal mapping, Computational design and fabrication, Discrete differential geometry

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1 INTRODUCTION

Net curves with constant angles arise in numerous applications, including geometric modeling, graph visualization, mechanical structures, and architecture. The locally uniform intersection angle between edges helps form aesthetically pleasing tracing curves and ensures relatively regular spacing of curve patterns—serving appearance design, force distribution, and cost-effective fabrication with repetitive beams or panels. This constant-angle property is a key factor for high-quality curve patterns on discrete surfaces, such as triangular meshes with constant 60° [Jiang et al. 2014a] or quad meshes with constant 90° [Bommes et al. 2013], though both impose severe limitations on surface shape and singular vertex configurations.

As quad meshes directly discretize smooth surfaces via curve network parametrization, they naturally connect to (discrete) differential geometry in analyzing local curvatures—making them our focus. Rather than enforcing constant 90° interior angles, orthogonality between two families of polylines is more broadly applicable across different curve nets, notably principal curvature nets (orthogonal conjugate nets). Two well-defined discrete principal curvature nets are *circular nets* [Bobenko and Suris 2008] and *conical nets* [Liu et al. 2006], where orthogonality emerges from geometric properties rather than from explicitly prescribed right angles at vertex stars—the latter proving computationally expensive [Dellinger et al. 2023; Wang et al. 2025].

Extending from right angles to constant acute or obtuse angles between quad mesh polylines yields the *discrete isogonal net*, offering greater design freedom and benefiting diverse applications integrating force, function, and fabrication. Despite surprisingly few contributions on isogonal net representation and construction,

existing gridshell structures built on surfaces with constant principal curvature ratios—discretized via isogonal nets—demonstrate advantages: straight or curved developable lamellae with identical repetitive nodes enable easy fabrication and assembly [Jiang et al. 2021b; Jimenez et al. 2020; Schling et al. 2023, 2022; Wang and Pottmann 2022]. Similarly, freeform architectural skins on affine linear Weingarten surfaces, also discretized via isogonal nets, achieve curvature-equivalent paneling that reduces costs through reusable molds [Pellis et al. 2020a,b].

Motivated by computational design and fabrication applications requiring cost-effective repetitive elements in architectural shell structures, we explore isogonal meshes by prescribing constant angles and local pattern similarity derived from checkerboard patterns controlled by edge ratios. This yields a framework grounded in curve net theory within discrete differential geometry, offering a large design space encoded by fast computational tools amenable to friendly fabrication.

1.1 Overview and contributions

The main goal of this paper is a careful study of isogonal quad mesh geometry and its applications in computational design and fabrication. After reviewing related work on mappings between discrete surfaces via checkerboard patterns (CBP) derived from control quad meshes—with applications to developable quad meshes—we present smooth theory for developable surfaces based on isogonal nets in Sec. 2, followed by discrete representations, constructions, and validations in Secs. 3–5, and design applications in Sec. 6.

- We investigate the geometry of smooth developable surfaces through isogonal Chebyshev nets, equivalent to isogonal geodesic nets in differential geometry (Sec. 2).
- Discrete developable surfaces are built from isogonal Chebyshev quads sampled along the diagonals of control quad meshes (DIC_{diag} -nets), which feature congruent parallelograms in generalized CBPs via edge-ratio regulation. By taking a fixed affine combination of the two diagonal meshes, this scheme yields equal-interior-angle (DI_{diag} -net) and similar (DIS_{diag} -net) parallelograms as discrete isogonal net models for the diagonal mesh alone or both control and diagonal meshes, respectively (Sec. 3).
- Developability of DIC_{diag} -nets is verified through well-defined normals, curve-like Gaussian images, and near-zero Gaussian curvature, with compatible rulings, shape exploration via affine maps and offsets (Sec. 5), and physical paper-cut validation (Sec. 6).
- The similar parallelogram property of DIS_{diag} -nets enables shape exploration through conformal maps, where parallelogram similarity induces similarity of area-maximizing inscribed and area-minimizing circumscribed ellipses (Sec. 4). DIS_{diag} -nets provide a rich shape space for modeling and pattern design on shell structures (Sec. 6).
- Both geometric representations and constraints on parallelogram shapes are explicit and quadratic in vertex variables, enabling an efficient computational tool for direct computation as well as numerically stable and fast-converging Gauss–Newton optimization (Sec. 6.5).

1.2 Previous work

1.2.1 Mappings between surfaces. Numerous geometry processing tasks require establishing correspondences and smooth mappings between geometric shapes, while typically maintaining topological connectivity. Common workflows include deforming an input surface toward a target configuration or constructing geometric relations between paired surfaces. Both tasks are generally formulated as numerical optimization problems defined on discrete mesh representations. Among various discrete mapping techniques, the checkerboard pattern derived from quad meshes serves as a versatile tool for conformal and isometric surface mappings.

1.2.2 Checkerboard pattern. Given a quad mesh, mid-edge subdivision can be performed by connecting the midpoints of all mesh edges. This process generates a distinctive CBP: every alternate face forms a parallelogram, whereas the remaining faces correspond to general spatial quads. These two types of quads are arranged in a staggered black-and-white checkerboard layout, forming a periodic tiling layout originally inspired by Olympic logo geometry, as first proposed in [Jiang et al. 2019b].



Follow-up studies have verified that the CBP extracted from a base quad mesh enables comprehensive analysis of the control mesh and its two associated diagonal meshes. Extensive research further demonstrates the broad advantages of CBP-based methods across disciplines. In discrete differential geometry, CBPs facilitate the construction of specialized discrete nets; in geometry processing, they support shape space exploration and cross-surface mapping; in architectural design, they empower patterned facade layouts and flexible shell structures. Representative applications cover orthogonal and isogonal net generation [Dellinger et al. 2023; Wang et al. 2025; Wang and Pottmann 2022], conformal mesh parameterization [Jiang et al. 2019b], isometric shape deformation [Inza et al. 2023; Jiang et al. 2020b, 2021a], transformable structural design [Jiang et al. 2020a, 2022], mechanical mechanism design [Jiang et al. 2024], custom architectural paneling [Pellis et al. 2020a], and lightweight gridshell fabrication [Jiang et al. 2021b].

1.2.3 Discrete developable surfaces. Developable surfaces fabricated from planar materials (bending, folding, rolling, cutting, assembling) are widely studied in computational design [Yuan et al. 2025]. Discrete approaches on triangle meshes include treating local polylines as hinge rulings [Stein et al. 2018], thinning Gaussian images via defined mesh normals [Binninger et al. 2021; Cheng et al. 2025], and converting height fields to nonsingular Hessians [Sellán et al. 2020].

For quad meshes, alternative strategies exist: developability via locally flat regions [Solomon et al. 2012], quad edges aligned with rulings [Liu et al. 2006], compatible locally defined rulings conjugate to tangent vectors [Inza et al. 2023], discrete orthogonal geodesic nets with local angle constraints [Rabinovich 2020], and geodesic parallel coordinates with controlled edge-length and angle constraints [Wang et al. 2019].

CBP-based methods [Jiang et al. 2019b] yield an orthogonal-diagonal mesh when inscribed parallelograms are rectangles; equal diagonal lengths produce squares. If these rectangles or squares are

congruent, every quad has identically long, orthogonally intersecting diagonals—constituting a novel discretization of developable surfaces, where congruent squares correspond to discrete orthogonal geodesic nets [Rabinovich 2020]. Isometric mappings preserving these congruent shapes enable intrinsic shape-space exploration and architectural fabrication [Jiang et al. 2020b, 2021a].

By generalizing the congruence condition from rectangles and squares to arbitrary parallelograms, we not only attain analogous parametrizations of developable surfaces but also establish a more comprehensive theoretical framework for characterizing isogonal nets with similar patterns.

2 SMOOTH DEVELOPABLE SURFACES VIA ISOGONAL CHEBYSHEV NETS

Prior to the discrete construction of developable surfaces—quad meshes whose inscribed parallelograms are congruent—we recall the smooth analogue: a Chebyshev net whose coordinate curves meet at a constant angle is developable.

Curve net. A surface is described by a two-parameter map $X : (u, v) \subset \mathbb{R}^2 \mapsto X(u, v) \subset \mathbb{R}^3$. Fixing $u = u_0$ gives the v -curves (tangent vector X_v); fixing $v = v_0$ gives the u -curves (tangent vector X_u). Together these two families form the (coordinate) curve net.

Isogonal net. Let $E = X_u^2$, $F = X_u \cdot X_v$, $G = X_v^2$ be the coefficients of the first fundamental form. The angle θ between the coordinate curves satisfies $\cos \theta = \frac{X_u \cdot X_v}{\|X_u\| \|X_v\|} = \frac{F}{\sqrt{EG}}$. If θ is constant (but not necessarily $\frac{\pi}{2}$) the net is termed *isogonal*, and $X(u, v)$ is an isogonal parametrization with $F/\sqrt{EG} = \text{const.}$

Chebyshev net. A parametrization $X(u, v)$ is a *Chebyshev net* if every coordinate quad has opposite sides of equal length, which is equivalent to the metric conditions $\|X_u\|_v = \|X_v\|_u = 0$, i.e. $E = E(u)$, $G = G(v)$. After a natural re-parametrization,

$$E = X_u^2 = 1, \quad F = X_u \cdot X_v = \cos \theta, \quad G = X_v^2 = 1. \quad (1)$$

The Gaussian curvature of a Chebyshev net is $K = -\frac{\theta_{uv}}{\sin \theta}$, where θ_{uv} denotes the mixed second partial derivative [do Carmo 1976]. If the net is isogonal, then θ is constant. Consequently, $\theta_{uv} = 0$, and hence $K = 0$. A surface with vanishing Gaussian curvature is developable; hence an isogonal Chebyshev net represents a developable surface.

Theorem 2.1. *An isogonal Chebyshev net is developable.*

The developability of an isogonal Chebyshev net aligns with that of an orthogonal geodesic net (*G-net*); in particular, an orthogonal *G-net* is simultaneously a Chebyshev net [Dellinger et al. 2023] and characterises a developable surface [Rabinovich 2020]. This smooth equivalence is summarized in the following propositions.

Proposition 2.2. *An isogonal Chebyshev net is geodesic (i.e. an isogonal G-net) and developable.*

PROOF. With the natural parametrisation Eq. (1) we have $\|X_u\| = \|X_v\| = 1$, so X_{uu} and X_{vv} coincide with the principal normal fields of the u - and v -curves, respectively; and satisfy $X_{uu} \cdot X_u = 0$ and $X_{vv} \cdot X_v = 0$. Differentiating F gives $X_{uu} \cdot X_v = -X_u \cdot X_{uv} = 0$ and $X_{vv} \cdot X_u = -X_v \cdot X_{uv} = 0$. Thus X_{uu} and X_{vv} are orthogonal to both tangent vectors X_u, X_v , i.e. parallel to the surface normal; consequently both coordinate families are geodesic.

An isogonal *G-net* represents a developable surface, which can be proved using the Gauss–Bonnet theorem:

$$\int_X K dA + \int_{\partial X} \kappa_g ds = 2\pi - \sum_{i=1}^4 \theta_i,$$

where the geodesic curvature vanishes along all four boundary curves ($\kappa_g = 0$), and the four intersection angles satisfy $\theta_1 = \theta_3 = \theta$ and $\theta_2 = \theta_4 = \pi - \theta$, forcing the Gaussian curvature $K = 0$ on X . $\square$

Remark. That the coordinate curves of an isogonal Chebyshev net are geodesic can follow from another fact that the geodesic curvatures of the u - and v -curves satisfy $\kappa_g^u = -\theta_u$, $\kappa_g^v = -\theta_v$, (see Appendix A.1 for the proof), where θ_u and θ_v are the first partial derivatives of θ . Hence, if the angle θ is constant, then $\theta_u = \theta_v = 0$, so $\kappa_g^u = \kappa_g^v = 0$, and both families of curves are geodesics.

Proposition 2.3. *An isogonal Chebyshev net and an isogonal G-net are equivalent and represent a developable surface.*

PROOF. ($\Rightarrow$) Isogonal Chebyshev net is an isogonal *G-net* (Prop. 2.2).

($\Leftarrow$) For the u -curves (v fixed), let s be arc length. The unit tangent is $\mathbf{t} = X_u/\sqrt{E}$. The geodesic condition implies that the principal normal of the curve is parallel to the surface normal, so $\frac{d\mathbf{t}}{ds} \perp X_v$. Since $\frac{d\mathbf{t}}{ds} = \frac{1}{E} \frac{\partial}{\partial u} \left(\frac{X_u}{\sqrt{E}} \right)$, orthogonality gives

$$\frac{\partial}{\partial u} \left(\frac{X_u \cdot X_v}{\sqrt{E}} \right) = 0.$$

Using $X_u \cdot X_v = F = c\sqrt{EG}$, where $c = \cos \theta$ is constant and $\theta \neq \frac{\pi}{2}$, then

$$\frac{\partial}{\partial u} (c\sqrt{G}) = c \frac{G_u}{2\sqrt{G}} = 0 \implies G_u = 0.$$

Interchanging the roles of u and v gives $E_v = 0$. Thus isogonal *G-net* is a Chebyshev net. $\square$

3 DISCRETE ISOGONAL NETS AND DEVELOPABLES

Building on the smooth theory of developable surfaces via isogonal Chebyshev nets, we examine their discrete counterparts realized by quad meshes whose congruent parallelograms form the underlying discrete diagonal structure. These quad meshes are further characterized as DIC_{diag} -nets, with the primary focus being the discussion and generalization of discrete isogonal nets as DI_{diag} -nets and DIS_{diag} -nets based on the CBP method (see Tab. 1, Figs. 2, 15).

3.1 Inscribed parallelograms prescribed by a parameter

Mid-edge subdivided parallelogram. Given a spatial quad, connecting the midpoints of its four consecutive edges yields an inscribed quad that is necessarily a parallelogram—the classical *Varignon's theorem* [Jiang et al. 2019b; Wikipedia 1731]. Its two pairs of opposite sides are parallel to the diagonals of the original quad and have lengths equal to one-half of those diagonals. In the graphics literature this parallelogram is commonly termed the *mid-edge subdivided parallelogram* associated with the quad; when the construction is applied to every quad face of a coarse quad mesh it produces a globally consistent subdivision that can be iteratively refined to improve shape quality.

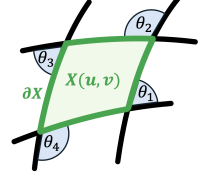


Table 1. Quad meshes characterizing different control and diagonal meshes via constrained $(t, \bar{t})$ -parallelograms as proposed in this paper.

	$\text{DI}_{\text{diag-net}}$	$\text{DIS}_{\text{diag-net}}$	$\text{DIC}_{\text{diag-net}}$
Representation	$\theta = \text{const.}$	$(\theta, \phi) = \text{const.} \iff (\theta, \frac{l_2^y}{l_1^y} = \frac{l_2^y}{l_1^y}) = \text{const.}$	$(\theta, \phi, c_1) = \text{const.} \iff (c_1, c_2, c_3) = \text{const.}$
Geometry (∇, ∇)	const. interior angles	similar	congruent
Equation(s)	Eq. (3) $\iff E_{\text{DI}}$	Eqs. (3,5) $\iff E_{\text{DIS}}$	Eq. (6) $\iff E_{\text{DIC}}$
Control mesh	general	orthogonal ($\phi = \frac{\pi}{2}$)	orthogonal ($\phi = \frac{\pi}{2}$)
Diagonal mesh	isogonal (θ)	isogonal (θ)	isogonal (θ) Chebyshev $\iff$ developable mesh

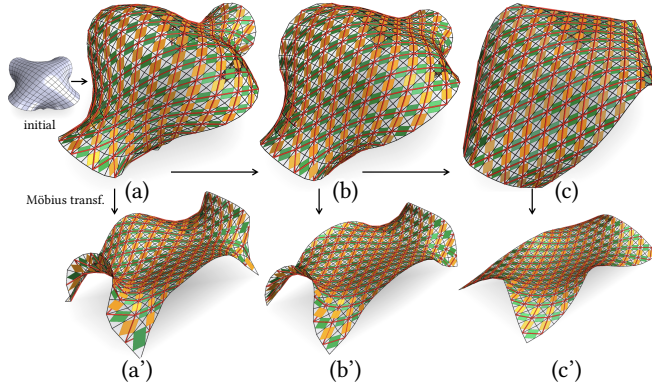


Fig. 2. (a–c) A general quad mesh is continuously optimized into three types of isogonal nets: $\text{DI}_{\text{diag-net}}$ (a), $\text{DIS}_{\text{diag-net}}$ (b), and $\text{DIC}_{\text{diag-net}}$ (c). These correspond respectively to: isogonal diagonal mesh (red); isogonal diagonal mesh paired with orthogonal control mesh; and orthogonal control mesh together with an isogonal Chebyshev (geodesic) diagonal mesh. The checkerboard quads in the three cases feature equal-interior-angle similar $(t, \bar{t})$ -parallelograms, similar $(t, \bar{t})$ -parallelograms, and congruent $(t, \bar{t})$ -parallelograms, respectively. (a'–c') Corresponding results under Möbius transformation: (a') and (b') retain the structure of $\text{DI}_{\text{diag-net}}$ and $\text{DIS}_{\text{diag-net}}$, while (c') preserves only the $\text{DIS}_{\text{diag-net}}$. This is because congruence—and consequently developability—is not preserved under Möbius transformation.

Edge-gliding inscribed parallelogram. For a spacial quad with edge vertices $\mathbf{v}_i, \mathbf{v}_j (j = i + 1 \bmod 4)$, the mid-point $\mathbf{p}_i$ on edge $\mathbf{v}_i \mathbf{v}_j$ is $\mathbf{p}_i = 0.5 \cdot (\mathbf{v}_i + \mathbf{v}_j)$, which corresponds to the ratio $t = 0.5$ in the convex combination

$$\mathbf{p}_i = t \cdot \mathbf{v}_i + (1 - t) \cdot \mathbf{v}_j, t \in [0, 1].$$

Geometrically, $\mathbf{p}_i$ is the interpolated point on edge $\mathbf{v}_i \mathbf{v}_j$ such that $\mathbf{v}_j - \mathbf{p}_i = t \cdot (\mathbf{v}_j - \mathbf{v}_i)$, i.e. the vector from $\mathbf{p}_i$ to $\mathbf{v}_j$ is parallel to the edge and its length equals fraction t of the edge length. Equivalently, $t = \frac{\|\mathbf{p}_i \mathbf{v}_j\|}{\|\mathbf{v}_i \mathbf{v}_j\|}$ and $1 - t = \frac{\|\mathbf{v}_i \mathbf{p}_i\|}{\|\mathbf{v}_i \mathbf{v}_j\|}$ from $\|\mathbf{v}_i \mathbf{v}_j\| = \|\mathbf{v}_i \mathbf{p}_i\| + \|\mathbf{p}_i \mathbf{v}_j\|$, so

$$\mathbf{p}_i = \frac{\|\mathbf{p}_i \mathbf{v}_j\|}{\|\mathbf{v}_i \mathbf{v}_j\|} \cdot \mathbf{v}_i + \frac{\|\mathbf{v}_i \mathbf{p}_i\|}{\|\mathbf{v}_i \mathbf{v}_j\|} \cdot \mathbf{v}_j.$$

Collecting the four vertices gives the compact relation

$$\begin{bmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \\ \mathbf{p}_4 \end{bmatrix} = \begin{bmatrix} t & 1-t & 0 & 0 \\ 0 & 1-t & t & 0 \\ 0 & 0 & t & 1-t \\ t & 0 & 0 & 1-t \end{bmatrix} \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3 \\ \mathbf{v}_4 \end{bmatrix}. \quad (2)$$

The mid-edge subdivided parallelogram appears when $t = 0.5$. More generally, for any $t \in [0, 1]$ the four points $\mathbf{p}_i$ satisfying Eq. (2)

glide counter-clockwise along the edges while maintaining the proportional splits $(1 - t, t, 1 - t, t)$, thereby generating the broader family of edge-gliding inscribed parallelograms. Similarly, the two pairs of opposite sides of these parallelograms remain parallel to the diagonals of the original quad and have lengths equal to $1 - t$ and t times those diagonals, respectively (see Fig. 3). For later discussion we give such an edge-gliding parallelogram a name.

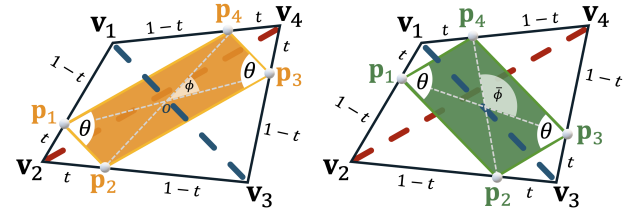


Fig. 3. Left: a t -parallelogram (yellow) inscribed in a quad $\mathbf{v}_1 \mathbf{v}_2 \mathbf{v}_3 \mathbf{v}_4$; its vertices divide the edges in the ratios $(1 - t, t, 1 - t, t)$. Right: the conjugate $\bar{t}$ -parallelogram (green) with ratios $(t, 1 - t, t, 1 - t)$. Both parallelograms have sides parallel to the quad diagonals and the same interior angle θ (the angle between the quad diagonals); their respective diagonal angles ϕ and $\bar{\phi}$ are in general different.

Definition 3.1 (∇, ∇). A t -parallelogram (∇) is a parallelogram whose four vertices glide counter-clockwise along the edges of a quad with proportional splits $(1 - t, t, 1 - t, t)$, $t \in [0, 1]$. Its conjugate, a $\bar{t}$ -parallelogram (∇), reverses the splitting order to $(t, 1 - t, t, 1 - t)$. The t - and $\bar{t}$ -parallelograms are abbreviated as $(t, \bar{t})$ -parallelograms.

Lemma 3.1. For any fixed quad, every inscribed t -parallelogram has identical interior angles.

PROOF. For every t -parallelogram, two sides are parallel to the diagonals of the quad. These sides form affine combinations of the two diagonal vectors, with affine ratios t and $1 - t$; hence, one pair of opposite interior angles equals θ —the fixed angle between these diagonals—while the other pair equals $\pi - \theta$. $\square$

Parallelogram shape. The shape of a parallelogram is determined by its interior angle θ and its diagonal angle ϕ , which can be expressed in terms of the side-length ratio λ and diagonal-length ratio μ [Wang et al. 2025]. Two parallelograms are similar if and only if both their interior angles and their diagonal angles coincide, which is equivalent to unique values of λ and μ . If, in addition, the corresponding side lengths are equal, the parallelograms are congruent.

Mirrored similarity. When the t -parallelogram and $\bar{t}$ -parallelogram share the same interior angles, the angles between their oriented

diagonal vectors are complementary (Fig. 4, bottom); hence one becomes a scaled mirror image of the other. We still consider them similar, and refer to them collectively as similar $(t, \bar{t})$ -parallelograms. This mirrored configuration distinguishes our work from the identically oriented parallelograms within each quad face in [Wang et al. 2025] (Fig. 4, top).

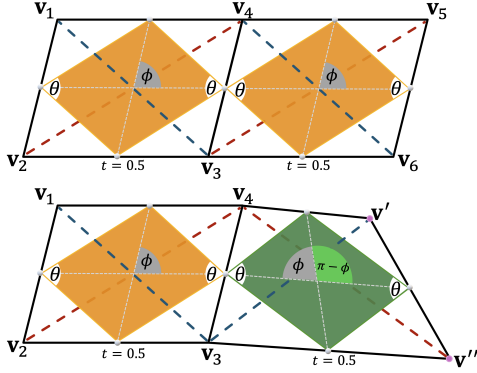


Fig. 4. Top: Discrete isogonal net [Wang et al. 2025] with similar mid-edge subdivided parallelograms having constant oriented interior angle θ and diagonal angle ϕ . Bottom: Our proposed DIS_{diag}-net and DIC_{diag}-net with $(t, \bar{t})$ -parallelograms in mirrored similarity, featuring constant interior angle θ and complementary diagonal angles ϕ and $\pi - \phi$.

3.2 CBP and two diagonal meshes of a control quad mesh
CBP prescribed by constant ratios. Given a quad control mesh, extending the mid-edge subdivision from the fixed ratio $t = 0.5$ to arbitrary prescribed ratios $(1 - t, t, 1 - t, t)$ with $t \in [0, 1]$ yields edge-gliding t -parallelograms on every second quad. The remaining quads are automatically inscribed by $\bar{t}$ -parallelograms with ratios $(t, 1 - t, t, 1 - t)$ taken counter-clockwise along their edges (Fig. 5).

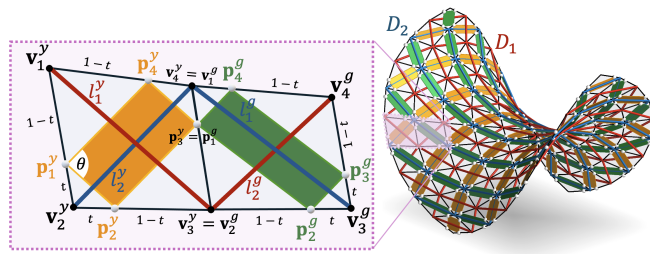


Fig. 5. Any quad mesh (black) carries a CBP obtained by subdividing each face: every second quad is a t -parallelgram (yellow) or its conjugate $\bar{t}$ -parallelgram (green), while the remaining faces (white) remain general spatial quads. The entire family of such CBPs is generated by the affine combination $tD_1 + (1 - t)D_2, t \in [0, 1]$, of the two diagonal meshes D_1 (red) and D_2 (blue).

Connecting these gliding (edge-inscribed) points produces a new quad mesh that retains the CBP: every second face is a parallelogram governed by the ratios $(1 - t, t, 1 - t, t)$ or $(t, 1 - t, t, 1 - t)$, while

the other faces are general spatial quads, exactly as in the classical mid-edge subdivision. The shape of the inscribed parallelograms determines the properties of the corresponding CBPs, as detailed in Tab. 1.

Two diagonal meshes. For a given control quad mesh C , an entire family of CBPs with parameter $t \in [0, 1]$ are obtained by inscribing a parallelogram in every face of C . Opposite sides of these t -parallelograms are parallel to the two diagonals of the corresponding control quad and have lengths $t\|\mathbf{d}_1\|$ and $(1 - t)\|\mathbf{d}_2\|$, where $\mathbf{d}_1 = \mathbf{v}_3^f - \mathbf{v}_1^f$ and $\mathbf{d}_2 = \mathbf{v}_4^f - \mathbf{v}_2^f$ are the diagonals of C . The collections of all such diagonals form two edge-disjoint quad meshes D_1 and D_2 , called the *diagonal meshes* of C (Fig. 5). These meshes are translation-equivalent and encode the same discrete geometry as C . Each CBP can be written as the single affine combination $tD_1 + (1 - t)D_2$, hence it is in one-to-one correspondence with the pair (D_1, D_2) .

First-order faces. Regarding C as a discrete surface, its edges are treated as discrete first-order derivatives. The diagonals of D_1 and D_2 play an identical role, yielding discrete derivatives along the diagonal directions. Every inscribed t - or $\bar{t}$ -parallelogram has four sides parallel to these diagonals; hence its edges are also discrete first-order derivatives. We therefore interpret the inscribed parallelograms as *first-order faces* of the discrete surface. The visual regularity of the polylines formed by consecutive parallelogram edges is thus governed by the fairness of the underlying D_1 and D_2 polylines.

Second-order derivatives. Since each t - or $\bar{t}$ -parallelogram vertex is a linear combination of the endpoints of its gliding edge, the difference of opposite vertices—i.e., the diagonal vectors of the parallelogram—can be interpreted as *second-order derivatives* evaluated at the barycenter of the four vertices, which coincides with the intersection of the two diagonals. As illustrated in Fig. 3-left, $\mathbf{p}_3 - \mathbf{p}_1 = t\mathbf{d}_1 + (1 - t)\mathbf{d}_2$, $\mathbf{p}_4 - \mathbf{p}_2 = -t\mathbf{d}_1 + (1 - t)\mathbf{d}_2$ and the barycenter

$$\begin{aligned} \mathbf{o} &= \frac{1}{4}(\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 + \mathbf{p}_4) = \frac{1}{2}[t\mathbf{v}_1 + (1 - t)\mathbf{v}_2 + t\mathbf{v}_3 + (1 - t)\mathbf{v}_4] \\ &= \frac{1}{2}(\mathbf{p}_1 + \mathbf{p}_3) = \frac{1}{2}(\mathbf{p}_2 + \mathbf{p}_4), \end{aligned}$$

reduces to the centroid $\frac{1}{4}(\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_4)$ of the control quad when $t = 0.5$.

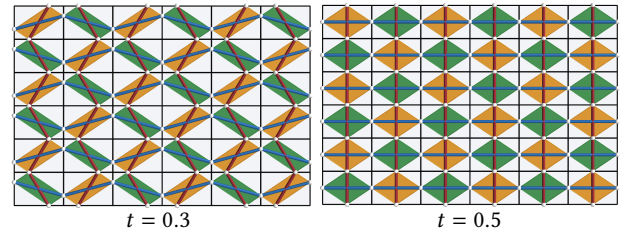


Fig. 6. The visual regularity of the polylines formed by consecutive diagonals of the inscribed parallelograms is governed by the ratio t . Left: For a generic $t \in [0, 1]$ ($t \neq 0.5$), the diagonal polylines of the inscribed parallelograms form a zigzag pattern relative to the fair control mesh C . Right: At $t = 0.5$ these polylines coincide with the medial lines of C , recovering the same isoline smoothness as C itself; the parallelograms become rhombi with orthogonal medial lines.

For a regular rectangular grid as control mesh, the medial lines (interpreted as second-order derivatives) yield constant angle $\phi = \frac{\pi}{2}$ —the diagonal angle of the mid-edge subdivided parallelograms ($t = 0.5$, congruent rhombi)—while the diagonal mesh has constant angle θ , the interior angle of these parallelograms (Fig. 6, right). When $t \neq 0.5$ (Fig. 6, left), the interior angle θ of the $(t, \bar{t})$ -parallelograms is preserved along the diagonal mesh, but their diagonal angle ϕ no longer coincides with the control-mesh angle ($\frac{\pi}{2}$). Instead, the oriented diagonals alternate between ϕ and $\pi - \phi$, giving the connected diagonal polylines a zigzag appearance.

3.3 DI_{diag} -net: quad mesh with constant diagonal angles

It is non-trivial to formulate the isogonality condition of a surface parameterization within the standard discrete framework when working with a single quad mesh. Analogous to the orthogonality exhibited by the diagonals of the control mesh [Jiang et al. 2019b], we define the isogonal diagonals of the control mesh following a comparable approach.

When every t - and $\bar{t}$ -parallelogram in the CBP carries the same interior angle, the angle between the two diagonals of each control quad is constant. These diagonals belong to two coarser diagonal meshes D_1 and D_2 (equal up to translation) and together constitute a refined diagonal mesh that is isogonal. To avoid confusion, we clarify that all diagonal meshes mentioned hereafter refer to this refined, isogonal mesh. This observation motivates the following definition.

Definition 3.2 (DI_{diag} -net). *A quad mesh whose checker-wise faces are inscribed with t - and $\bar{t}$ -parallelograms sharing a common interior angle θ is called a discrete I_{diag} -net (or DI_{diag} -net for short); its diagonal mesh is isogonal.*

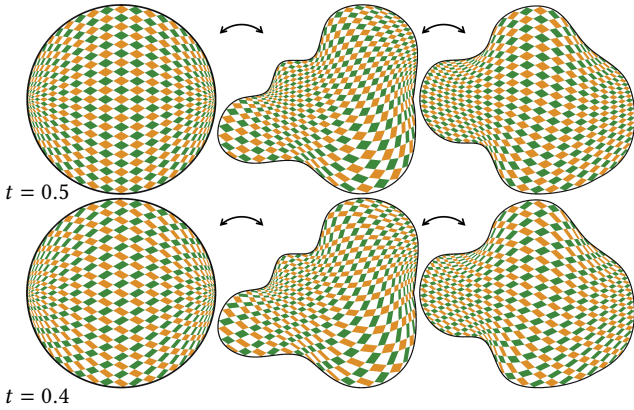


Fig. 7. Conformal mappings between DI_{diag} -nets with different planar boundaries; all parallelograms have constant interior angles $(\frac{\pi}{3}, \frac{2\pi}{3})$. Different CBPs extracted at $t = 0.5$ (top) and $t = 0.4$ (bottom).

The DI_{diag} -net coincides with the isogonal meshes of [Jiang et al. 2021b] and has been used to construct Weingarten meshes (e.g., minimal meshes) by imposing isogonal and asymptotic (planar vertex star) conditions simultaneously. The special case $t = 0.5$ yields the classical mid-edge subdivision CBP (Fig. 7, top), while general

ratios (e.g., $t = 0.4$) produce CBPs with alternating parallelogram orientations (Fig. 7, bottom).

For a DI_{diag} -net, all corresponding interior angles of the $(t, \bar{t})$ -parallelograms are equal (Lem. 3.1). By varying $t \in [0, 1]$ we obtain CBPs with different $(t, \bar{t})$ -parallelograms shapes; this directly changes the diagonal angle ϕ of the parallelograms (Fig. 8 and 15- (a_1-d_1)).

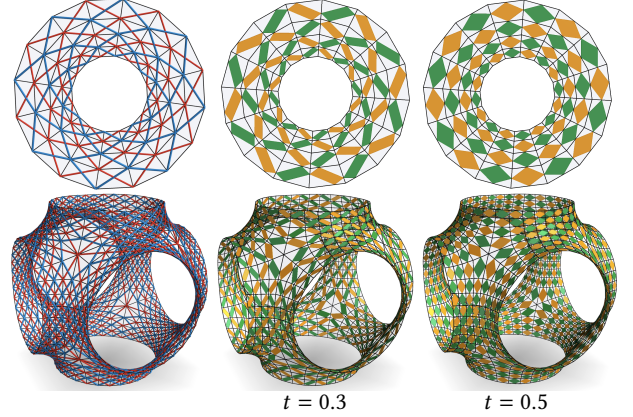


Fig. 8. CBP configurations with different $(t, \bar{t})$ -parallelogram shapes. Top: 2D DI_{diag} -net; bottom: 3D DI_{diag} -net. Both diagonal meshes (red and blue) have constant angle $\theta = \frac{\pi}{3}$. All parallelograms in each CBP share the same interior angles $(\frac{\pi}{3}, \frac{2\pi}{3})$ but may differ in shape due to the different angle ϕ .

Geometric representations. A DI_{diag} -net is characterized by a constant interior angle θ shared by every $(t, \bar{t})$ -parallelograms. This is encoded by fixing the angle between the diagonals of each control quad (Fig. 5):

$$\frac{(\mathbf{v}_3^f - \mathbf{v}_1^f) \cdot (\mathbf{v}_4^f - \mathbf{v}_2^f)}{\|\mathbf{v}_3^f - \mathbf{v}_1^f\| \cdot \|\mathbf{v}_4^f - \mathbf{v}_2^f\|} = \cos \theta, \quad (3)$$

where f indexes every quad face with vertices $\mathbf{v}_1^f, \dots, \mathbf{v}_4^f$. Introducing diagonal lengths l_1^f, l_2^f and unit directions $\mathbf{u}_1^f, \mathbf{u}_2^f$ gives

$$\begin{cases} \mathbf{v}_3^f - \mathbf{v}_1^f = l_1^f \mathbf{u}_1^f, \mathbf{v}_4^f - \mathbf{v}_2^f = l_2^f \mathbf{u}_2^f, \\ \|\mathbf{u}_1^f\| = 1, \|\mathbf{u}_2^f\| = 1, \\ \mathbf{u}_1^f \cdot \mathbf{u}_2^f = \cos \theta. \end{cases} \quad (4)$$

3.4 DIS_{diag} -net: quad mesh with similar $(t, \bar{t})$ -parallelograms

As shown in [Wang et al. 2025], the control mesh and its diagonal mesh are simultaneously isogonal precisely when all mid-edge subdivided parallelograms are similar (Fig. 4, top). The diagonal mesh has constant interior angle θ , while the control mesh has constant diagonal angle ϕ .

In this section, we extend DI_{diag} -nets to DIS_{diag} -nets whose parallelograms are similar in mirrored configuration (Fig. 4, bottom). Unlike [Wang et al. 2025], this setting yields: (i) a rich family parameterized by t ; and (ii) an isogonal diagonal mesh with constant θ paired with an orthogonal control mesh having $\phi = \frac{\pi}{2}$. Our proposed DIS_{diag} -nets and DIC_{diag} -nets feature orthogonal control meshes

with similar or congruent rhombi at $t = 0.5$, and diverse similar or congruent parallelograms at $t \neq 0.5$ (Fig. 15).

For a regular rectangular grid as control mesh (Fig. 6), the angle θ between diagonals of control quads is constant and independent of t . The angle between diagonals of parallelograms alternates between ϕ and $\pi - \phi$ for adjacent quads, and equals $\frac{\pi}{2}$ at $t = 0.5$ where the parallelograms become rhombi; the medial lines then achieve the same smoothness as the isolines, yielding a medial-line mesh that approximates C up to translation (Fig. 6, right).

Extending the regular rectangular grid to a general quad mesh with a CBP of similar $(t, \bar{t})$ -parallelograms, the mirrored similarity of t - and $\bar{t}$ -parallelograms preserves the isogonal condition. This observation motivates the following definition.

Definition 3.3 ($\text{DIS}_{\text{diag-net}}$). A quad mesh whose checker-wise faces are inscribed with **similar** $(t, \bar{t})$ -parallelograms generates a $\text{DIS}_{\text{diag-net}}$, denoted a discrete $\text{IS}_{\text{diag-net}}$, abbreviated $\text{DIS}_{\text{diag-net}}$.

Geometric representation. Similarity of every $(t, \bar{t})$ -parallelogram is encoded by constant angles θ and ϕ (or $\pi - \phi$), equivalently determined by unique side-ratio λ and diagonal-ratio μ (Fig. 5):

$$\begin{cases} \lambda = \frac{\|\mathbf{P}_4^y - \mathbf{P}_1^y\|}{\|\mathbf{P}_2^y - \mathbf{P}_1^y\|} = \frac{1-t}{t} \frac{\|\mathbf{v}_4^y - \mathbf{v}_1^y\|}{\|\mathbf{v}_3^y - \mathbf{v}_1^y\|} = \frac{1-t}{t} \frac{l_2^y}{l_1^y}, & (\angle) \\ \lambda = \frac{\|\mathbf{P}_2^g - \mathbf{P}_1^g\|}{\|\mathbf{P}_4^g - \mathbf{P}_1^g\|} = \frac{1-t}{t} \frac{\|\mathbf{v}_3^g - \mathbf{v}_1^g\|}{\|\mathbf{v}_4^g - \mathbf{v}_1^g\|} = \frac{1-t}{t} \frac{l_1^g}{l_2^g}, & (\nabla) \\ \mu = \frac{\|\mathbf{P}_3^f - \mathbf{P}_1^f\|}{\|\mathbf{P}_4^f - \mathbf{P}_2^f\|}. & (\text{all quads}) \end{cases}$$

Here l_1^y, l_2^y and l_1^g, l_2^g are the diagonal lengths of the two alternating checker quads, respectively. Requiring the same λ for every quad forces the checker diagonals to satisfy

$$\frac{l_2^y}{l_1^y} = \frac{l_1^g}{l_2^g}, \quad (5)$$

a condition independent of t .

The global diagonal ratio μ is automatically fixed by the constant interior angle θ and side ratio λ derived from [Wang et al. 2025]:

$$\cos \theta = \frac{(1 + \lambda^2)(\mu^2 - 1)}{2\lambda(1 + \mu^2)} \implies \mu^2 = \frac{\lambda^2 + 2\lambda \cos \theta + 1}{\lambda^2 - 2\lambda \cos \theta + 1},$$

and the diagonal angle ϕ simplifies to

$$\cos \phi = \frac{\lambda^2 - 1}{\sqrt{(\lambda^2 - 1)^2 - 4\lambda^2 \cos^2 \theta}}.$$

Thus, the similarity of all $(t, \bar{t})$ -parallelograms is equivalent to the constant diagonal-angle condition (Eq. (3)) combined with equal checkerboard diagonal-length ratios (Eq. (5)).

Corollary 3.1.1. A $\text{DIS}_{\text{diag-net}}$ is a quad mesh whose $(t, \bar{t})$ -parallelograms are similar, governed by Eqs. (3) and (5); varying t produces the entire family of such CBPs. Different t values yield different similarity classes of $(t, \bar{t})$ -parallelograms.

A simple $\text{DIS}_{\text{diag-net}}$ with constant non-right angles can be extracted from a planar rectangular grid and mapped to curved planar $\text{DIS}_{\text{diag-net}}$ s under conformal maps such as $1/z, z^2, e^z$ (Fig. 9).

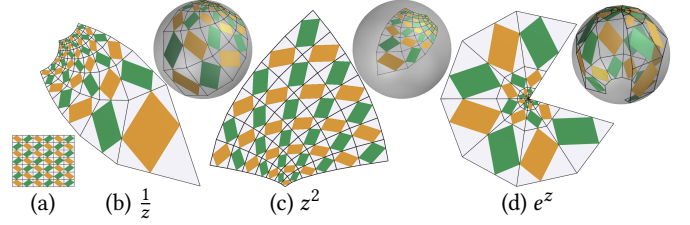


Fig. 9. A CBP extracted from a planar rectangular grid (a) at $t = 0.4$ preserves the similarity of the inscribed parallelograms under the conformal maps $1/z$ (b), z^2 (c), e^z (d) in the plane and their inverse stereographic projections onto the unit sphere.

Shape encoding and special choices. In a $\text{DIS}_{\text{diag-net}}$, the family of similar $(t, \bar{t})$ -parallelograms is governed by two global invariants: the side-ratio λ and the diagonal-ratio μ (Fig. 10). These invariants yield the specializations (Fig. 11)

- rhombus: $\lambda = 1$ (or $\phi = \frac{\pi}{2}$); orthogonal control mesh;
- rectangle: $\mu = 1$ (or $\theta = \frac{\pi}{2}$); orthogonal diagonal mesh;
- square: $\lambda = \mu = 1$ (or $\theta = \phi = \frac{\pi}{2}$);
- θ -parallelogram ($\theta = \phi \neq \frac{\pi}{2}$): $\lambda = \mu \neq 1$ (or $\cos \theta = \frac{\lambda^2 - 1}{2\lambda}$).

Corollary 3.1.2. Within a fixed $\text{DIS}_{\text{diag-net}}$ ($\theta \neq \frac{\pi}{2}$) the similarity class is fixed by the local ratio $r_0 = \frac{l_2^y}{l_1^y}$ (Fig. 11,12):

- rhombus: $t = \frac{r_0}{1+r_0}$;
- θ -parallelogram: $t = \frac{r_0}{r_0 + \cos \theta + \sqrt{\cos^2 \theta + 1}}$.

PROOF. The t -parallelogram is a rhombus iff $\lambda = \frac{1-t}{t} \frac{l_2^y}{l_1^y} = 1$, giving $t = \frac{l_2^y}{l_1^y + l_2^y}$. The second formula follows from solving $\cos \theta = \frac{\lambda^2 - 1}{2\lambda}$ and is equivalent to

$$t = \frac{\|\mathbf{d}_2^y\|^2}{\|\mathbf{d}_2^y\|^2 + \mathbf{d}_1^y \cdot \mathbf{d}_2^y + \sqrt{\|\mathbf{d}_1^y\|^2 \|\mathbf{d}_2^y\|^2 + (\mathbf{d}_1^y \cdot \mathbf{d}_2^y)^2}},$$

with $\mathbf{d}_1^y = \mathbf{v}_3^y - \mathbf{v}_1^y$, $\mathbf{d}_2^y = \mathbf{v}_4^y - \mathbf{v}_2^y$. By Eq. (5) the complementary $\bar{t}$ -parallelograms adopt the same shape automatically; at $t = 0.5$ the second expression yields $(l_2^f)^2 - (l_1^f)^2 = 2l_1^f l_2^f$ as in [Wang et al. 2025], where l_1^f, l_2^f denote the diagonal lengths of the control quad. $\square$

Eq. (5) requires any two neighboring quads to have reciprocal ratios between their oriented diagonal lengths, equivalent to the reciprocal side ratios λ and $1/\lambda$ of mid-edge subdivided parallelograms touching at the midpoint of their shared edge. These ratios λ and $1/\lambda$ influence the diagonal angles ϕ and $\pi - \phi$ of a parallelogram, exhibiting mirrored similarity with fixed interior angle θ (Fig. 4). Treating the diagonal vectors of parallelograms as second-order derivatives, both diagonal angles ϕ and $\pi - \phi$ are interpreted as angles of the control mesh. For a mesh with relatively fair polyline smoothness, these angles must balance at $\frac{\pi}{2}$, consistent with the regular grid case in 2D (Fig. 6). Thus, the parallelograms reduce to rhombi with orthogonal diagonals and equal side lengths ($\phi = \frac{\pi}{2}$, $\lambda = r_0 = 1$, $t = 0.5$), and the control mesh becomes orthogonal—numerically

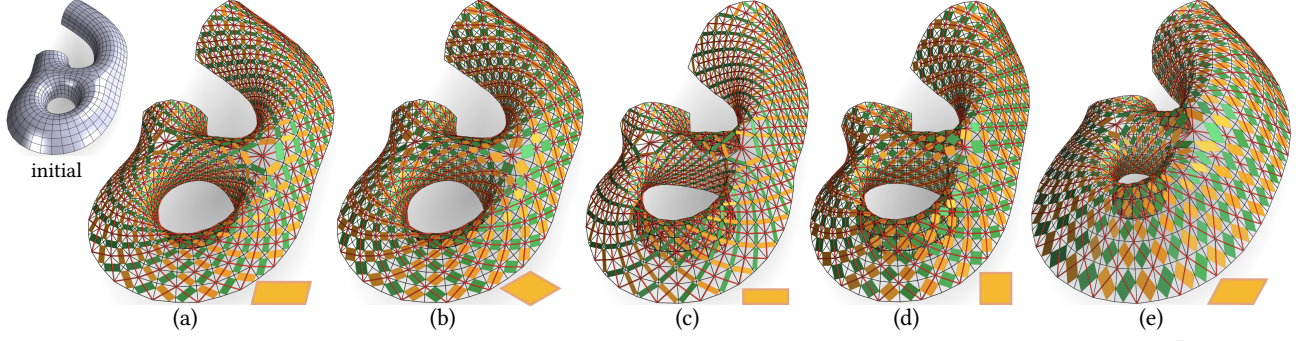


Fig. 10. A general quad mesh optimized DISdiag-nets with similar (a) parallelograms ($t = 0.3$), (b) rhombi, (c) rectangles, (d) squares and (e) $\frac{\pi}{3}$ -parallelograms.

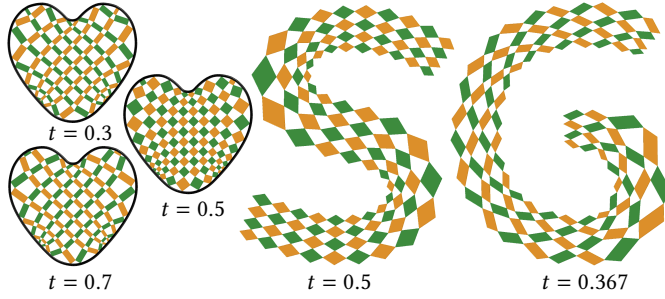


Fig. 11. Various 2D shapes textured by similar $(t, \bar{t})$ -parallelograms. For the heart shape, similar rectangles appear at $t = 0.3$ and $t = 0.7$, and similar squares at $t = 0.5$. The S and G shapes use similar rhombi at $t = 0.5$ and similar 55° -parallelograms at the computed value $t = 0.367$.

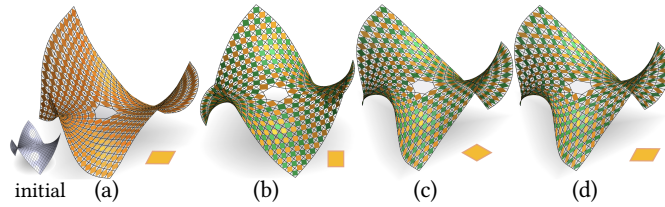


Fig. 12. General quad mesh optimized: (a) DI-net with $\frac{\pi}{3}$ -parallelograms; (b) DISdiag-net with squares (orthogonal control and diagonal meshes); (c-d) DISdiag-nets with $\theta = \frac{\pi}{3}$, where similar rhombi (c) and $\frac{\pi}{3}$ -parallelograms (d) are extracted via computed t . Control meshes of DISdiag-nets (b-d) are orthogonal due to $\lambda \approx 1$; similar rhombi occur at $t \approx 0.5$.

validated in Figs. 12, 13. In Fig. 13, 2D and 3D quad meshes satisfying only Eq. (5) are orthogonal, characterized by mid-edge subdivided rhombi, with the maximum relative deviation (MRD) of the diagonal angle ϕ from $\frac{\pi}{2}$ being 9.69×10^{-5} and 3.90×10^{-4} , respectively.

Corollary 3.1.3. *The control mesh of a DISdiag-net is orthogonal, and the mid-edge subdivided parallelograms are similar rhombi.*

PROOF. For all quads, Eq. (5) yields the convex energy

$$\begin{aligned} E(x) &= \sum_{t-\square} (\|l_2^y\|^2 - \|l_1^y\|^2)^2 + \sum_{\bar{t}-\square} (\|l_2^g\|^2 - \|l_1^g\|^2)^2 \\ &= (x^2 - 1)^2 \sum_{t-\square} \|l_1^y\|^4 + \left(\frac{1}{x^2} - 1\right)^2 \sum_{\bar{t}-\square} \|l_1^g\|^4. \end{aligned}$$

Since both coefficient sums are strictly positive, $E(x)$ is minimized when $x^2 - 1 = \frac{1}{x^2} - 1$, giving $x = 1$. Thus the parallelograms become rhombi with equal diagonals, and the control mesh is orthogonal. $\square$

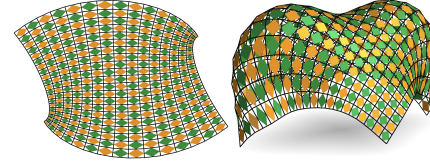


Fig. 13. 2D and 3D quad meshes satisfying only Eq. (5) become orthogonal via mid-edge subdivided rhombi.

Relation with discrete orthogonal nets [Dellinger et al. 2023]. The orthogonal control mesh property of DISdiag-nets follows from Eq. (5), equivalent to the equal-diagonal representation of discrete orthogonal structures in [Dellinger et al. 2023]. This agrees with the intuitive conclusion from mirrored symmetry and is validated numerically; a rigorous proof within (discrete) differential geometry remains for future work.

Relation with DI-net [Wang et al. 2025]. A DISdiag-net combines an orthogonal control mesh with isogonal diagonals (Cor. 3.1.3), independent of t . At $t = 0.5$ it specializes to an orthogonal DI-net due to mirrored similarity of $(t, \bar{t})$ -parallelograms. For any fixed t , all inscribed $(t, \bar{t})$ -parallelograms remain similar in mirrored configuration, whereas DI-nets present similarly oriented parallelograms only at $t = 0.5$. Varying t in a DISdiag-net changes parallelogram shapes (Cor. 3.1.1) while preserving similarity, thus enlarging the pattern space (Figs. 12, 14).

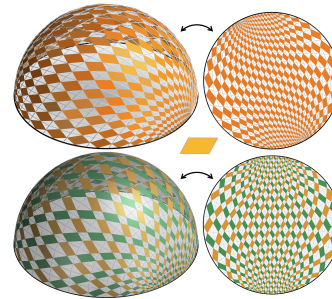


Fig. 14. Quad meshes composed of similar $\frac{\pi}{3}$ -parallelograms, where DISdiag-nets (bottom) generate the CBPs of DI-nets (top) via $t \neq 0.5$, and conformal mappings between spherical patches (left) and their circular images (right) preserve such $\frac{\pi}{3}$ -parallelogram structures.

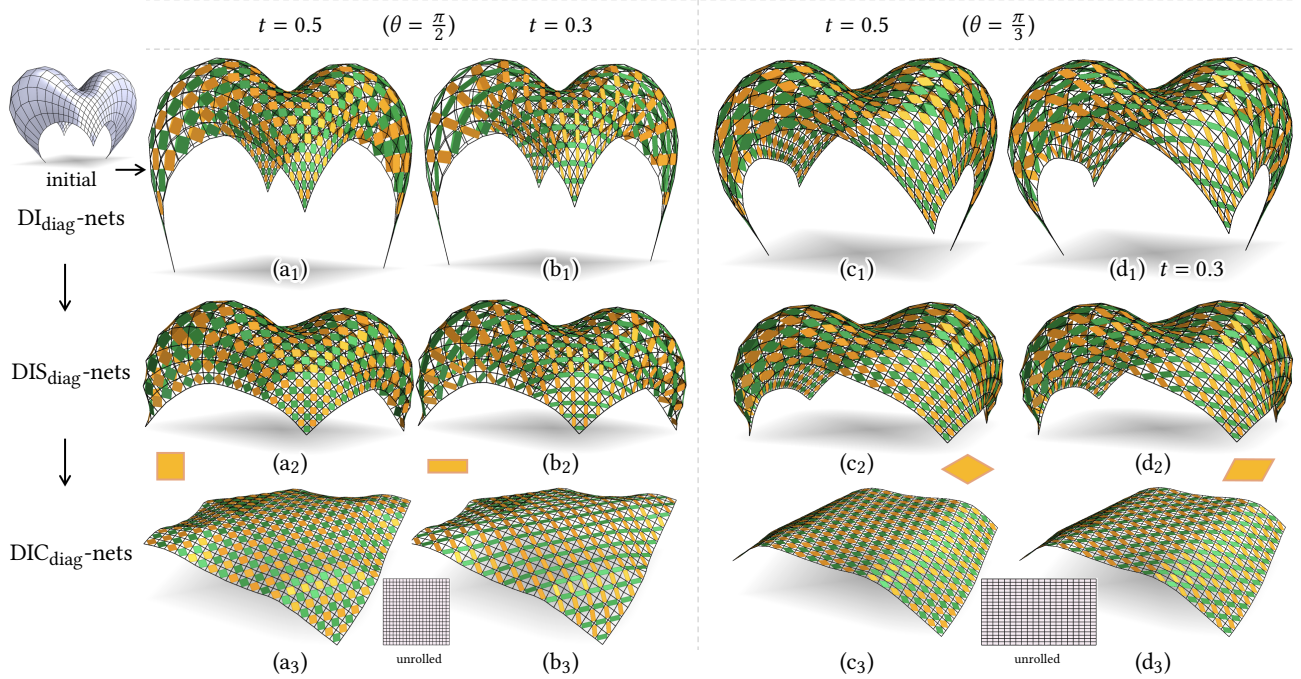


Fig. 15. Special $(t, \bar{t})$ -parallelogram shapes on gradually optimized DI_{diag} -nets (a_1 – d_1), DIS_{diag} -nets (a_2 – d_2), and DIC_{diag} -nets (a_3 – d_3). Visualized at $t = 0.5$ (a_1 – a_3), $t = 0.3$ (b_1 – b_3) for $\theta = \frac{\pi}{2}$, and $t = 0.5$ (c_1 – c_3), $t = 0.3$ (d_1), θ -parallelograms (d_2 – d_3) for $\theta = \frac{\pi}{3}$. Similar and congruent squares, rectangles, rhombi, and $\frac{\pi}{3}$ -parallelograms appear for DIS_{diag} -nets and DIC_{diag} -nets. The developments of DIC_{diag} -nets with $\theta = \frac{\pi}{2}$ and $\theta = \frac{\pi}{3}$ yield square and rectangular grids, respectively, demonstrating their developable quality.

3.5 DIC_{diag} -net: quad mesh with congruent $(t, \bar{t})$ -parallelograms

When every $(t, \bar{t})$ -parallelogram in the CBP is congruent, the DIS_{diag} -net possesses not only constant interior and diagonal angles but also constant side lengths of the parallelograms. Such a net is defined as follows.

Definition 3.4 (DIC_{diag} -net). A DI_{diag} -net whose corresponding checker-wise $(t, \bar{t})$ -parallelograms are **congruent** is called a discrete IC_{diag} -net, abbreviated DIC_{diag} -net.

A DIC_{diag} -net, defined as a quad mesh with congruent $(t, \bar{t})$ -parallelograms, is a special case of a DIS_{diag} -net. The hierarchy of DI_{diag} -nets is now established as follows (see Tab. 1, Fig. 2 and 15):

- A DI_{diag} -net requires only that every $(t, \bar{t})$ -parallelogram has a common interior angle θ .
- A DIS_{diag} -net further fixes the diagonal angle ϕ or $\pi - \phi$ of each parallelogram, so all $(t, \bar{t})$ -parallelogram are similar.
- A DIC_{diag} -net additionally requires all edge lengths of these parallelograms to be equal, yielding congruent parallelogram.

A DIC_{diag} -net makes both the CBP and the associated diagonal mesh discrete Chebyshev nets [Bobenko and Suris 2008] up to global scaling factors $1/t$ or $1/(1-t)$, since every quad face has four edges of equal length. This observation yields:

Theorem 3.2. A quad mesh whose checkerboard faces are inscribed with congruent $(t, \bar{t})$ -parallelograms—a DIC_{diag} -net—forms a discrete isogonal Chebyshev net in the diagonal direction and thus discretizes a developable surface.

For a DIC_{diag} -net, the control mesh and diagonal mesh represent the same discrete developable surface, yielding a discrete counterpart to the smooth theory in Thm. 2.1. The diagonal mesh is a discrete isogonal Chebyshev net; for fair polylines it behaves approximately like a discrete G-net, becoming one in the smooth limit (consistent with Props. 2.2 and 2.3).

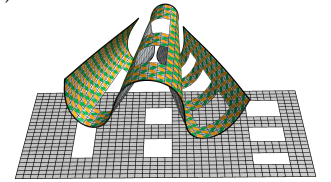
DIC_{diag} -nets preserve the orthogonal control mesh property from Eq. (5). At $t = 0.5$, DIC_{diag} becomes a special DI -net with congruent rhombi—a case not treated in [Wang et al. 2025]. This intrinsic property allows any regular part of a DIC_{diag} -net to unroll into a regular rectangular grid (see inset and Fig. 15), with congruent rhombi at $t = 0.5$ and congruent parallelograms at $t \neq 0.5$.

Corollary 3.2.1. Regular regions of a DIC_{diag} -net are isometric to a regular rectangular grid.

Since polylines of a regular rectangular grid are straight orthogonal lines in the plane, polylines of regular regions in a DIC_{diag} -net are geodesic and orthogonal.

Corollary 3.2.2. A DIC_{diag} -net is locally an orthogonal geodesic net (DOG net).

Thus the control mesh of a DIC_{diag} -net is a DOG net, and the diagonal net is a discrete isogonal net. Their four families of polylines form a 4-web of geodesics with constant angles between each pair—a special case of 3-web or 4-web theory [Schling et al. 2022].



Geometric representations. For the $(t, \bar{t})$ -parallelograms to be congruent across the CBP, both their angles and edge lengths must be uniform. The two side edges of every parallelogram are parallel to the control-quad diagonals $\mathbf{v}_3^f - \mathbf{v}_1^f$ and $\mathbf{v}_4^f - \mathbf{v}_2^f$; hence congruence reduces to keeping (Fig. 5)

$$\begin{cases} \|\mathbf{v}_3^y - \mathbf{v}_1^y\| = c_1, & \|\mathbf{v}_4^y - \mathbf{v}_2^y\| = c_2, & (\swarrow), \\ \|\mathbf{v}_3^g - \mathbf{v}_1^g\| = c_2, & \|\mathbf{v}_4^g - \mathbf{v}_2^g\| = c_1, & (\searrow), \\ (\mathbf{v}_3^f - \mathbf{v}_1^f) \cdot (\mathbf{v}_4^f - \mathbf{v}_2^f) = c_3 & \text{(all quads)}. \end{cases} \quad (6)$$

constant for every t - or $\bar{t}$ -parallelogram. The sole distinction between t and $\bar{t}$ cases is the swapped order of c_1 and c_2 (see Fig. 16). The interior angle θ , consistent with Eq. (3) satisfies

$$c_3 = c_1 c_2 \cos \theta. \quad (7)$$

The DIC_{diag} -net is independent of the parameter t ; for any fixed t all inscribed $(t, \bar{t})$ -parallelograms are congruent.

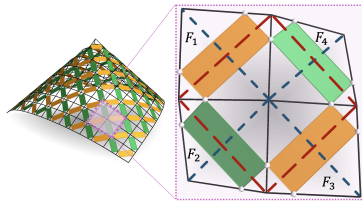


Fig. 16. Four congruent parallelograms inscribed in adjacent quads F_1 – F_4 produce a central diagonal quad (red) with equal edge lengths and four outgoing diagonals (blue) of identical length around the vertex star.

The same classification holds for congruent parallelograms in a DIC_{diag} -net (Fig. 15-(a3–d3)):

Corollary 3.2.3. *The congruent $(t, \bar{t})$ -parallelograms of a DIC_{diag} -net become:*

- rhombi if $\lambda = 1 \Leftrightarrow \phi = \frac{\pi}{2} \Leftrightarrow t = 0.5 \Leftrightarrow c_1 = c_2$;
- rectangles if $\mu = 1 \Leftrightarrow \theta = \frac{\pi}{2} \Leftrightarrow c_3 = 0$;
- squares if $\lambda = \mu = 1 \Leftrightarrow \theta = \phi = \frac{\pi}{2}$;
- θ -parallelograms if $\lambda = \mu \neq 1 \Leftrightarrow \theta = \phi \neq \frac{\pi}{2}$

$$\Leftrightarrow \begin{cases} t^2 c_1^2 - (1-t)^2 c_2^2 + 2t(1-t)c_3 = 0, & (\swarrow) \\ (1-t)^2 c_1^2 - t^2 c_2^2 - 2t(1-t)c_3 = 0. & (\searrow) \end{cases}$$

where λ and μ denote the side-length and diagonal-length ratios, and θ and ϕ are the interior and diagonal angles of the parallelograms.

Relation with congruent mid-edge rectangles/squares. DIC_{diag} -net generalizes the orthogonal Chebyshev net result of [Jiang et al. 2019b]. There, discrete developable surfaces are encoded by quad meshes whose congruent mid-edge rectangles (or squares) yield an orthogonal diagonal mesh; squares give orthogonality for both the diagonal and the control mesh. Developability is argued by exhibiting an isometric mapping to a planar regular grid, but no curvature-based proof is supplied. DIC_{diag} -net replaces rectangles/squares with congruent parallelograms and produces an isogonal diagonal mesh; the orthogonal case is recovered at $\theta = \frac{\pi}{2}$ (see Fig. 17). Thm. 3.2 therefore extends the orthogonal-Chebyshev-net statement to isogonal Chebyshev nets and provides a unified, curvature-based proof via Thm. 2.1.

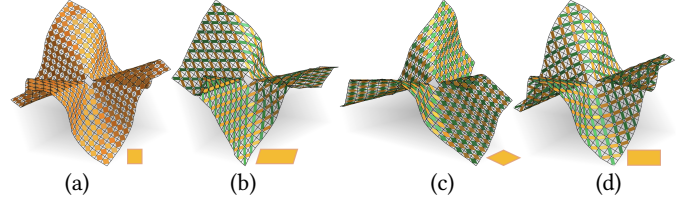


Fig. 17. Developable quad meshes: (a) with congruent mid-edge squares [Jiang et al. 2019b]; (b–d) DIC_{diag} -nets with congruent parallelograms ($\theta = \frac{\pi}{3}, t = 0.3$), rhombi ($\theta = 130^\circ$), and rectangles via the proposed method.

4 CONSTRUCTION OF DIS_{diag} -NETS

Compared to DI_{diag} -nets, DIS_{diag} -nets impose additional angle constraints to ensure parallelogram similarity. While any regular surface admits orthogonal parametrizations [do Carmo 1976], and isogonal nets can be achieved locally by affine maps of these, DI_{diag} -nets are thus freely available on discrete regular surfaces. This section therefore focuses on constructing the more constrained DIS_{diag} -nets.

Conformal mapping is a fundamental geometric operation. Conformal transformations preserve local angles between tangent curves on surfaces while altering global geometric configurations, without changing the underlying manifold structure.

DIS_{diag} -nets are conformal invariants. The constant-angle property between control and diagonal meshes makes DIS_{diag} -nets naturally related to conformal principal curvature nets (*isothermal nets*) and constructible via conformal mappings—such as stereographic projection between sphere and plane, and Möbius transformations in 3D. With tunable parameters, diverse DIS_{diag} -net geometries can be rapidly generated and transformed.

4.1 DIS_{diag} -net from isothermal nets

Isothermal parametrizations exist on any regular surface; they are locally conformal to the plane and, by composition, to any other surface. An *isothermal net* consists of conformal principal curvature lines with first fundamental form $E = G$ and $F = 0$. Geometrically, small squares in the parameter domain map to square-shaped quads on the surface.

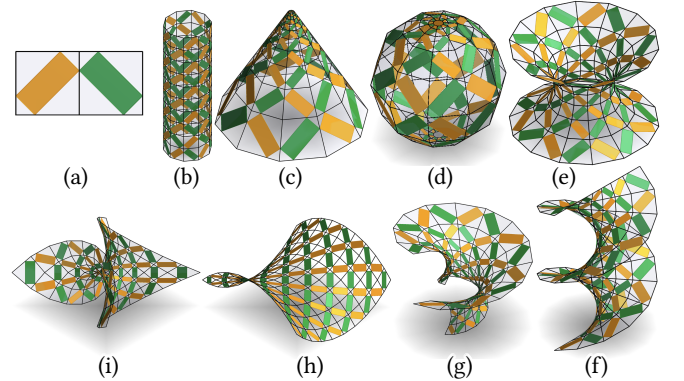


Fig. 18. (a) $(t, \bar{t})$ -rectangles from adjacent squares. (b–i) DIS_{diag} -nets with similar $(t, \bar{t})$ -rectangles on isothermal nets: (b) cylinder, (c) cone, (d) sphere, (e) catenoid, (f) helicoid, (g) a conjugate minimal surface combined by catenoid and helicoid, (h) Enneper surface, (i) Scherk surface.

The diagonals of each such square are orthogonal vectors of equal length; thus the extracted $(t, \bar{t})$ -parallelograms are similar rectangles (squares when $t = 0.5$). Fig. 18 shows DIS_{diag} -nets with similar rectangles derived from classical isothermal parametrizations.

Remark. The similar parallelograms of DI -nets in [Wang et al. 2025] are special mid-edge subdivided parallelograms with diagonal angle $\frac{\pi}{4}$, extracted from one family of principal curvature lines and one family of diagonal lines of an isothermal net; however, the $(t, \bar{t})$ -parallelograms extracted for $t \neq 0.5$ are no longer similar.

4.2 DIS_{diag} -nets via stereographic projection

The *stereographic projection* is a conformal map from the sphere S^2 (excluding the north pole) to the plane. Consider the sphere $x^2 + y^2 + (z - 1)^2 = 1$ centered at $(0, 0, 1)$ with radius 1, with north pole $N = (0, 0, 2)$.

Forward projection $\Phi : S^2 \setminus \{N\} \rightarrow \mathbb{R}^2$ sends a point $p = (x, y, z)$ to the intersection of line with the xy -plane (Fig. 19):

$$\Phi(x, y, z) = (u, v) = \left(\frac{2x}{2-z}, \frac{2y}{2-z} \right).$$

Inverse projection $\Phi^{-1} : \mathbb{R}^2 \rightarrow S^2 \setminus \{N\}$ recovers the spherical point from (u, v) (Fig. 19):

$$\Phi^{-1}(u, v) = \left(\frac{4u}{u^2 + v^2 + 4}, \frac{4v}{u^2 + v^2 + 4}, \frac{2(u^2 + v^2)}{u^2 + v^2 + 4} \right).$$

With these explicit formulas, spherical and planar DIS_{diag} -nets can be interchanged: given either, the other is obtained by applying Φ or Φ^{-1} .

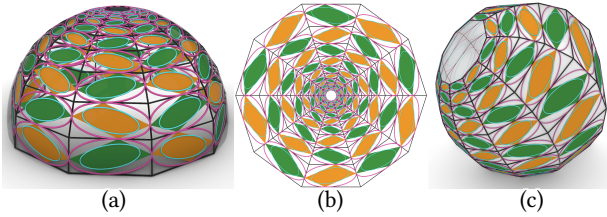


Fig. 19. (a) Spherical DIS_{diag} -net with similar parallelograms and circum-/inscribed ellipses. (b) Stereographic projection onto the plane. (c) 3D Möbius transformation. Both preserve parallelogram similarity and ellipse shapes.

4.3 DIS_{diag} -net via Möbius transformation

The *Möbius transformation* [Ahlfors 1981] is a fundamental class of analytic conformal mappings. Its standard planar form is the fractional linear transformation

$$w(z) = \frac{az + b}{cz + d}, \quad ad - bc \neq 0,$$

which decomposes into translations, rotations, similarities, and inversions. This transformation preserves angles strictly, is invertible, and realizes conformal mappings of isogonal nets in both 2D and 3D.

Applying Möbius transformations to prescribed DIS_{diag} -nets yields diverse surface shapes, aiding shape exploration and providing good initializations for further optimization with different constant angles. Fig. 2-(b') and 19-(c) illustrates how 3D Möbius transformations preserve parallelogram similarity.

The conformality is visualized through preservation of parallelogram similarity, whose shape is prescribed by two parameters (interior angle θ and diagonal angle ϕ). In the next section we show that this similarity can alternatively be prescribed by a single parameter—the eccentricity of area-maximizing inscribed or area-minimizing circumscribed ellipses. Fig. 19-(c) demonstrates that both parallelogram similarity and ellipse shapes remain invariant under this wider class of conformal maps.

4.4 Validation of conformal mapping by ellipse packing

Similar parallelograms in the CBP can be utilized as pattern designs, adorned with inscribed and circumscribed ellipses. For a parallelogram, multiple inscribed ellipses tangent to its four edges and multiple circumscribed ellipses passing through its four vertices exist. The inscribed ellipse with the maximal area is tangent to the midpoints of the edges [Salmon 1900], while the circumscribed ellipse with the minimal area aligns its conjugate diameters with the diagonals of the parallelogram. These ellipses and parallelograms can be treated as affine mapped results from the inscribed and circumscribed ellipses of rectangles, with their barycenters coinciding (see inset and Appendix A.2 for the construction).

For all similar parallelograms, the ratio $\frac{a}{b}$ remains constant, where a and b are the semi-major and semi-minor axis lengths, respectively. Thus the ellipses have constant eccentricity

$$e = \sqrt{1 - \frac{b^2}{a^2}}.$$

Since ellipse shape is uniquely determined by eccentricity, all ellipses for similar parallelograms are geometrically similar.

Proposition 4.1. *A DIS_{diag} -net possesses similar area-maximizing inscribed ellipses and area-minimizing circumscribed ellipses for its similar $(t, \bar{t})$ -parallelograms.*

Neighboring circumscribed ellipses of a DIS_{diag} -net touch at mesh-edge points dividing edges in ratios t and $1 - t$. Parallelogram similarity is thus prescribed via the eccentricities of these ellipses.

Fig. 20 shows a planar regular grid optimized as DI_{diag} -nets (b_1, b_2) or DIS_{diag} -nets (c_1, c_2, d) approximating a given freeform dome. DI_{diag} -nets (b_1, b_2) are constrained only by constant angle $\theta = \frac{\pi}{3}$; same-angle but non-similar parallelograms appear at $t = 0.5$ (b_1), with their inscribed and circumscribed ellipses shown in (b_2). The circumscribed ellipse colors indicate relative deviation from mean (RDM) of eccentricities, measuring similarity quality. In contrast, DIS_{diag} -net (c_2) exhibits low RDM due to parallelogram similarity. (d) shows similar inscribed and circumscribed ellipses for all similar $\frac{\pi}{3}$ -parallelograms, presenting a pattern design covering the freeform dome.

5 VERIFYING DEVELOPABILITY OF DIC_{diag} -NETS

As isometric to a regular rectangular grid (Cor. 3.2.1), a patch-shaped DIC_{diag} -net in 3D can be unrolled into a rectangular grid without stretching, only bending. The grid becomes square if the congruent rectangles of the CBP have $\theta = \frac{\pi}{2}$, as visualized in Fig. 15.

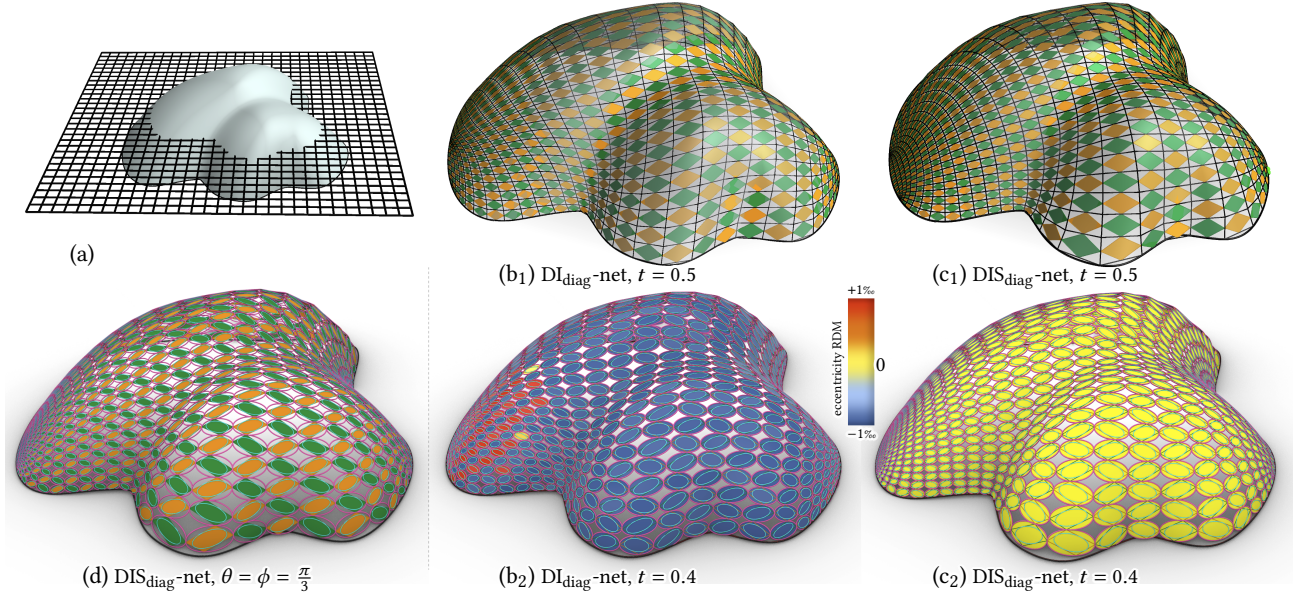
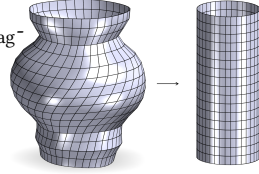


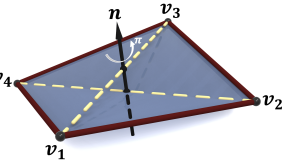
Fig. 20. (a) Reference surface and initial regular grid. Optimized DI_{diag} -net (b₁, b₂) and DIS_{diag} -net (c₁, c₂, d) with $\theta = \frac{\pi}{3}$. (b₁) Same-angle parallelograms at $t = 0.5$; (b₂) inscribed and circumscribed ellipses at $t = 0.4$. (c₁) Similar rhombi at $t = 0.5$; (c₂) inscribed and circumscribed ellipses at $t = 0.4$. (b₂, c₂) Similarity quality measured by RDM of circumscribed-ellipse eccentricities: -1% (blue) to $+1\%$ (red), yellow = zero. The DIS_{diag} -net (c₂) shows low RDM compared to the DI_{diag} -net (b₂). (d) Similar $\theta = \phi = \frac{\pi}{3}$ parallelograms with inscribed and circumscribed ellipses covering the reference surface, presenting a pattern design.

For a freeform mesh with rotational topology, optimization as a DIC_{diag} -net transforms it into a cylinder with regularly spaced orthogonal parameter lines; the cylinder can then be unrolled to a rectangular grid after cutting along a ruling.



5.1 Computing normals, Gauss images and curvatures

A Chebyshev quad is symmetric under a π rotation about the line through the midpoints of its diagonals [Bobenko and Suris 2008]. Computationally, opposite edges have equal lengths: $\|v_1 v_2\| = \|v_3 v_4\|$ and $\|v_2 v_3\| = \|v_1 v_4\|$, yielding two pairs of congruent triangles $\triangle v_1 v_3 v_4 \cong \triangle v_3 v_1 v_2$ and $\triangle v_4 v_1 v_2 \cong \triangle v_2 v_4 v_3$. The vector $(v_1 + v_3)/2 - (v_2 + v_4)/2$ connecting the diagonal midpoints is orthogonal to both diagonals. Thus this line serves as the Chebyshev quad normal.



In a DIC_{diag} -net, Chebyshev quads are associated with vertices; these lines define vertex normals, which determine rulings and whose unit vectors on the unit sphere form the curve-like image, i.e. *Gauss image* (Fig. 21–24).

The quality of the normals influence the Gauss image (Fig. 21). The fairness of normals in [Inza et al. 2023] is defined as $n_i - 2n_j + n_k = 0$ for three successive unit face normals. However, this expression forces these vectors to coincide due to their unit lengths. Only along rulings do the tangent-plane normals exhibit this property; for arbitrary polylines (even rulings after isometric deformation)

this is over-constrained and degrades the quality of normals and developable surfaces.

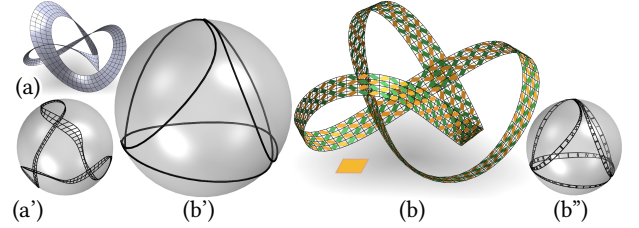


Fig. 21. An optimized developable strip from (a) to (b) via DIC_{diag} -net with congruent $\frac{\pi}{3}$ -parallelograms. (a') Gauss image with spherical area. (b') Vertex-defined normals yield thinned 1D curves on the sphere, outperforming face-defined normals (b'') from [Inza et al. 2023; Jiang et al. 2020b].

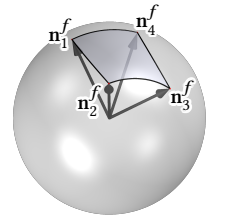
Instead, 1-dimensional Gauss image can be enforced by vanishing oriented areas for spherical quad faces from vertex normals $n_1^f, n_2^f, n_3^f, n_4^f$ of any control quad. The unit normal n at a regular vertex v with four neighbors v_i ($i = 1, \dots, 4$) is orthogonal to both diagonals $v_3 - v_1$ and $v_4 - v_2$, satisfying

$$\|n\|^2 = 1, \quad n \cdot (v_3 - v_1) = 0, \quad n \cdot (v_4 - v_2) = 0. \quad (8)$$

Thus, a set of curves on the unit sphere can be represented using the four unit normals attached to the four vertices of each quad, via parallel diagonal vectors:

$$(n_3^f - n_1^f) \times (n_4^f - n_2^f) = 0. \quad (9)$$

Compared with other discrete developable constructions—(a) discrete orthogonal geodesic net (DOG) [Rabinovich 2020], (b) discrete



geodesic parallel coordinates (DGPC) with controlled strip widths [Wang et al. 2019], and (d) ruling conjugacy method [Inza et al. 2023]—our $\text{DIC}_{\text{diag-net}}$ (c) achieves superior developability as assessed by Gauss images, where the nearly vanishing spherical areas indicate excellent curve-like quality (Fig. 22). This performance stems from our vertex normal fairness condition (Eq. (9)), which provides more geometric flexibility.

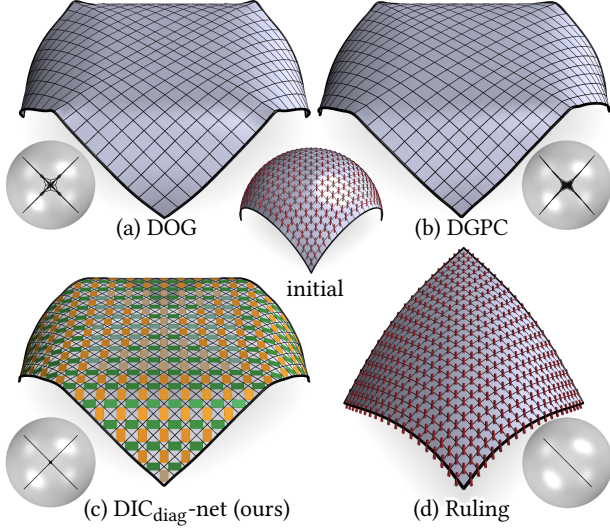


Fig. 22. An initial spherical patch optimized to developable quad meshes via (a) DOG [Rabinovich 2020], (b) DGPC [Wang et al. 2019], (c) our $\text{DIC}_{\text{diag-net}}$, and (d) ruling conjugacy [Inza et al. 2023]. Gauss images assess developability: our method achieves nearly vanishing spherical areas due to vertex normal fairness (Eq. (9)); the ruling method (d) performs well but depends heavily on initial rulings.

The discrete shape operator σ , which maps tangent planes of a quad mesh to its Gauss image on the unit sphere, can be defined using vertex-based Chebyshev quad normals instead of face normals [Jiang et al. 2020b]:

$$\sigma : \begin{cases} \mathbf{v}_3 - \mathbf{v}_1 \mapsto -(\mathbf{n}_3 - \mathbf{n}_1), \\ \mathbf{v}_4 - \mathbf{v}_2 \mapsto -(\mathbf{n}_4 - \mathbf{n}_2), \end{cases}$$

satisfying the symmetry condition

$$(\mathbf{n}_3 - \mathbf{n}_1) \cdot (\mathbf{v}_4 - \mathbf{v}_2) = (\mathbf{v}_3 - \mathbf{v}_1) \cdot (\mathbf{n}_4 - \mathbf{n}_2)$$

at each vertex star $[\mathbf{v}, \mathbf{v}_1, \dots, \mathbf{v}_4]$ with normals $[\mathbf{n}, \mathbf{n}_1, \dots, \mathbf{n}_4]$. Following the smooth theory, the Gaussian curvature K is then given by

$$K(\mathbf{v}) = \det(\sigma) = \frac{(\mathbf{n}_3 - \mathbf{n}_1) \times (\mathbf{n}_4 - \mathbf{n}_2) \cdot \mathbf{n}}{(\mathbf{v}_3 - \mathbf{v}_1) \times (\mathbf{v}_4 - \mathbf{v}_2) \cdot \mathbf{n}}.$$

The vanishing Gaussian curvature characteristic of developable surfaces is numerically verified in Figs. 23 and 24.

5.2 Visualizing rulings

For a developable surface, one family of principal curvature lines are straight lines—*rulings*—whose tangent planes are constant. The ruling $\mathbf{r}$ is conjugate to any tangent vector $\mathbf{e}$ of any surface curve, equivalent to $\Pi(\mathbf{r}, \mathbf{e}) = 0$ where Π is the second fundamental form.

A method to compute rulings for discrete developable quad meshes uses this conjugacy relation [Inza et al. 2023]. Along a mid-line of quad faces, mid-edge parallelograms serve as tangent contact elements with discrete surface normals. Intersections of neighboring tangent elements yield rulings conjugate to the mid-line tangent. Each contact element has two mid-line tangents, giving four rulings from four neighboring quads. These ruling fields must be consistent: two pairs of averaged rulings along each mid-line are parallel to ensure the degenerate conjugacy relation. Compatible rulings are optimized to obtain a discrete developable quad mesh.

Unlike this method where rulings are defined at mid-edge parallelograms (including surface normals), our rulings are defined at vertices, consistent with Chebyshev quad normals formed by four edge midpoints. Four rulings arise from intersections of the tangent plane at $\mathbf{v}$ with neighboring planes (blue, inset) at $\mathbf{v}_i$ with normals $\mathbf{n}_i$; rulings $\mathbf{r}_i$ anchored at edge midpoints are parallel to $\mathbf{n}_1 \times \mathbf{n}$, $\mathbf{n}_2 \times \mathbf{n}$, $\mathbf{n} \times \mathbf{n}_3$, $\mathbf{n} \times \mathbf{n}_4$ respectively. The degenerate conjugacy condition $(\mathbf{r}_1 + \mathbf{r}_3) \times (\mathbf{r}_2 + \mathbf{r}_4) = 0$ yields the ruling vector $\mathbf{r}$ at $\mathbf{v}$ satisfying

$$\|\mathbf{r}\|^2 = 1, \quad \mathbf{r} \cdot \mathbf{n} = 0, \quad \mathbf{r} \cdot (\mathbf{n}_3 - \mathbf{n}_1) = 0, \quad \mathbf{r} \cdot (\mathbf{n}_4 - \mathbf{n}_2) = 0. \quad (10)$$

Rulings are computed as $\mathbf{n} \times [(\mathbf{n}_3 - \mathbf{n}_1) \pm (\mathbf{n}_4 - \mathbf{n}_2)]$ depending on the orientation of $\mathbf{n}_3 - \mathbf{n}_1$ and $\mathbf{n}_4 - \mathbf{n}_2$, compatible with flat areas where rulings vanish (Fig. 23).

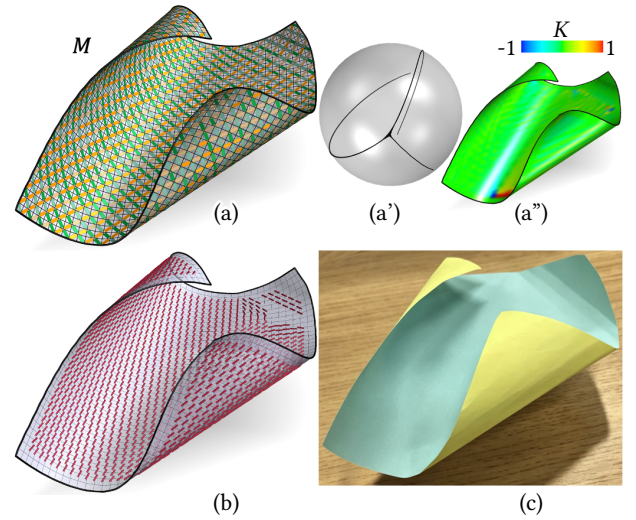


Fig. 23. (a) Discrete developable via $\text{DIC}_{\text{diag-net}}$ with CBP at $t = 0.3$, exhibiting developability through piecewise curves in Gauss image (a') and vanishing Gaussian curvature K (a''), green is zero). (b) Ruling vector fields computed from vertex normals. (c) Physical model bent from a 13:14 rectangular paper sheet, matching our result.

Discrete developable quad meshes can also be achieved by directly optimizing the conjugate ruling condition using our ruling expression Eq. (10) with well-defined vertex normals Eq. (8), rather

than applying DIC_{diag} -net conditions. Ruling-based remeshing follows [Verhoeven et al. 2022]. However, such methods [Inza et al. 2023] strongly depend on initial rulings for given freeform quad meshes; different initial rulings influence deformation directions (Fig. 22). Ruling-based methods may be explored in future work to discuss the shape space of developable meshes navigated by ruling vector fields.

5.3 Via affine maps and offsets

Developability is lost under Möbius transformations (Fig. 2) yet preserved under affine mappings and surface offsets.

Affine transformations, which combine translation, rotation, scaling, and shearing, disrupt the isogonal net property, resulting in dissimilar mid-edge parallelograms and non-rectangular planar unfoldings. Nevertheless, developability remains intact due to curve-shaped Gauss images with near-zero Gaussian curvature, allowing flexible shape exploration while fully retaining mesh topology.

Surface offsets are generated by displacing mesh vertices along their normals at a fixed distance with unchanged connectivity. A mesh and its offset share identical vertex normals and, accordingly, the same Gauss image. For developable meshes, vertex normals utilize Chebyshev quads, so the offset is developable. However, self-intersections may arise at large offset distances, requiring careful magnitude control.

Affine mapping and offsetting extend the 3D design space but introduce twisted unfoldings, which are undesirable in orthogonal geodesic nets [Rabinovich 2020] and isometric optimization for high-quality planar and 3D developable quad meshes [Jiang et al. 2020b]. Fig. 24 illustrates two discrete developable surfaces generated by applying the affine transformation and offsetting operations to the developable mesh M shown in Fig. 23.

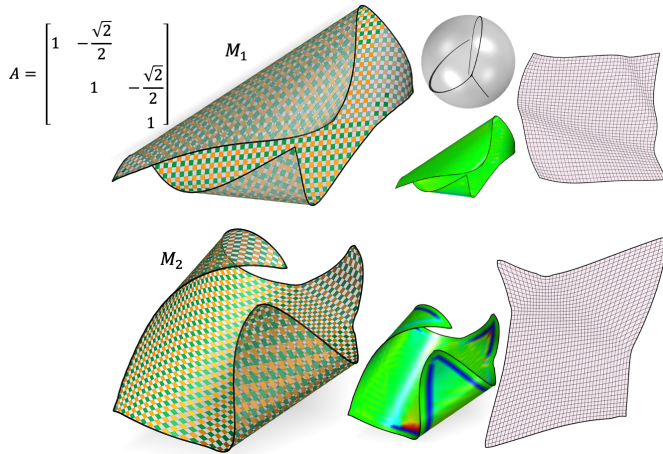


Fig. 24. Application of affine transformation A and offsetting to developable mesh M (Fig. 23) generates distinct morphologies M_1 and M_2 . Developability is retained via 1D Gaussian images and near-zero Gaussian curvature, while the isogonal property is compromised, as indicated by non-uniform parallelograms and distorted planar unfoldings.

6 COMPUTATIONAL DESIGN

6.1 Pattern design

Lamp shade. A rectangle maps to a parallelogram via an affine transformation. Thus a decorated carving pattern with rectangular frame can be sheared to a parallelogram frame (see inset). Similarity among all parallelogram panels facilitates rapid, uniform fabrication of carving patterns. A quad mesh with similar parallelograms can therefore be decorated with identical patterns for artistic effect. In Fig. 25, a lamp model is designed from a DIS_{diag} -net approximating a given freeform shape; all panels carry similar carving patterns that cast intricate shadows.

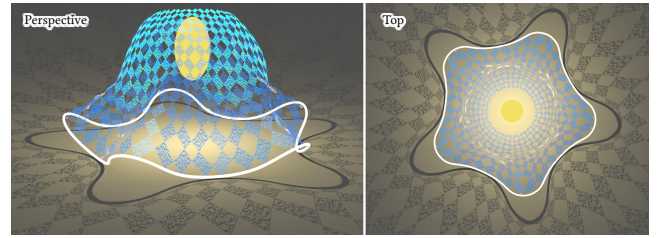


Fig. 25. Lamp shade decorated with similar parallelogram carving patterns, creating special shadow effects.

Quad meshes with singularities. A DI_{diag} -net requires an isogonal diagonal mesh, which typically demands regular or even-valence singular vertices to trace two families of diagonal polylines (Fig. 8). DIS_{diag} -nets inherit this topological constraint for their similar $(t, \bar{t})$ -parallelograms and alternating checker patterns (Figs. 12, 10, 28, 30). However, our half-edge-based representation of $(t, \bar{t})$ -parallelograms enables extraction at varying t , accommodating complex topologies including odd-valence singularities (Figs. 26, 16). Figure 26 shows a quad patch with four 3-valent singular vertices optimized as DIS_{diag} -nets with $\theta = 100^\circ$ and $\theta = 80^\circ$, with boundary vertices gliding on a plane. Despite unavoidable checker pattern imbalances along some edges (blue regions), both results at $t = 0.6$ exhibit similar parallelograms, nearly well-spaced orthogonal control polylines (black), and "isogonal" diagonal polylines. Thus DIS_{diag} -nets can handle complex quad meshes while yielding appealing CBPs composed of similar parallelograms, representing a viable architectural paneling system as illustrated in Fig. 27.

6.2 Ellipse packing

Packing structures endow surfaces with distinctive patterns—circles, spheres [Schiftner et al. 2009], ellipsoids [Xu et al. 2018], polyhedral cells [Pottmann et al. 2015]—yielding visual appeal in architectural design and cost-effective support structures, while revealing deep connections to conformal geometry and numerical optimization. These works emphasize the interplay between geometry, optimization, statics, and manufacturing, integrating form, function, and fabrication into unified design tools.

Even though [Xu et al. 2018] generalizes isotropic sphere packing to anisotropic ellipsoids, better adapting to local surface features

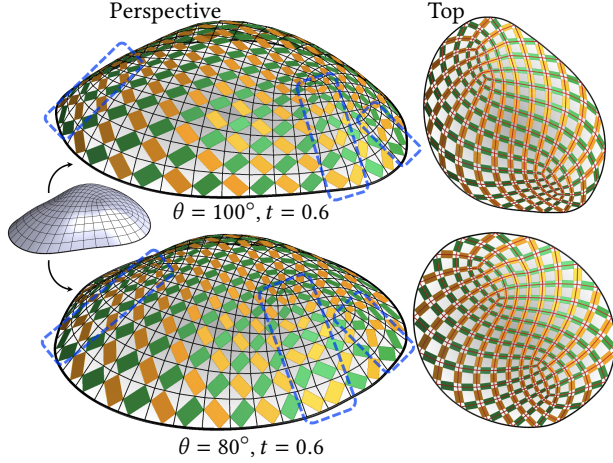


Fig. 26. A quad mesh with singularities optimized as DIS_{diag} -nets with $\theta = 100^\circ$ and $\theta = 80^\circ$, boundary vertices gliding on a plane. Despite unavoidable incompatible checker areas (blue), $(t, \bar{t})$ -parallelograms at $t = 0.6$ remain similar, yielding attractive patterns for architectural domes.

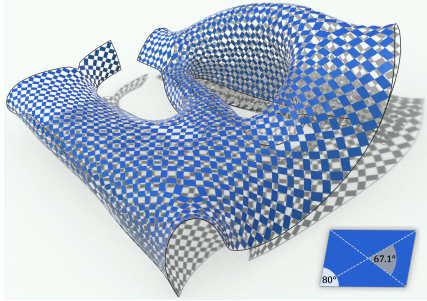


Fig. 27. Architectural paneling composed of 2191 similar parallelograms with similarity ratios in the range $[1 : 4.49]$. The CBP is extracted at $t = 0.6$ from the optimized DIS_{diag} -net with $\theta = 80^\circ$.

characterized by principal curvature differences, this enables fabrication of anisotropic regions using fewer elongated ellipsoids rather than many small spheres. However, these ellipsoids vary in shape, offering only visual design possibilities without fabrication-aware consistency.

Our work on DIS_{diag} -nets addresses this limitation by establishing that similar $(t, \bar{t})$ -parallelograms possess similar area-maximizing inscribed ellipses and area-minimizing circumscribed ellipses (Prop. 4.1). This result unifies parallelogram similarity with ellipse packing: a single eccentricity parameter prescribes shape across the entire surface, as seen in Figs. 19, 20, 28, and 29.

Shells with similar ellipses. Both sets of ellipses form ellipse packings on the DIS_{diag} -net. Packing of similar parallelograms or ellipses that exhibit appealing anisotropic patterns can be derived to benefit pattern design tasks on freeform surfaces [Jiang et al. 2015, 2014b; Pottmann et al. 2015; Xu et al. 2018], as seen in Fig. 28.

Orientation. DIS_{diag} -nets with different angle pairs (θ, ϕ) exhibit distinct global shapes while maintaining parallelogram similarity, owing to the orientation of angle formation. Fig. 29 illustrates this on a Möbius-transformed helicoid: (a) shows a DIS_{diag} -net with $(\theta = \frac{\pi}{3})$ -parallelograms decorated with inscribed ellipses; (b) shows the surface with $(\theta = \frac{2\pi}{3})$ -parallelograms decorated with circumscribed ellipses. Despite the contrasting visual appearance, both

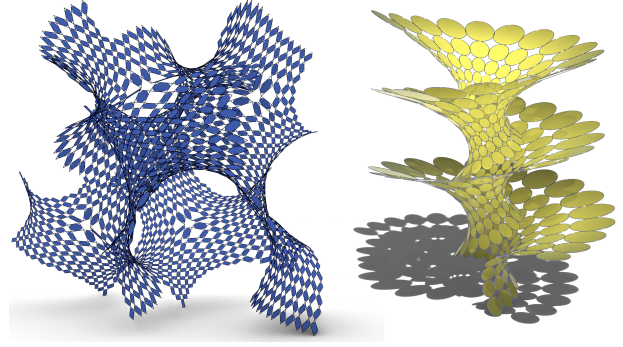


Fig. 28. Shell structures optimized as DIS_{diag} -nets are decorated with similar inscribed ellipse packing (left) and circumscribed packing (right) of the similar parallelograms.

nets share identical similarity classes of parallelograms and ellipses.

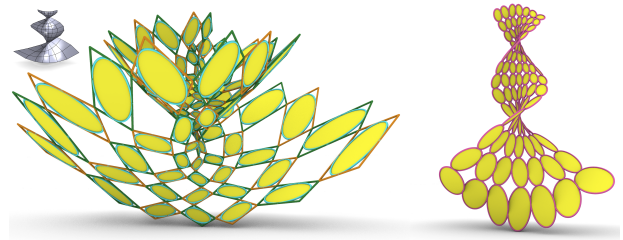


Fig. 29. A Möbius-transformed helicoid optimized as a DIS_{diag} -net with similar $\frac{\pi}{3}$ -parallelograms (left) and similar $\frac{2\pi}{3}$ -parallelograms (right); the former shows similar inscribed ellipses together with parallelogram frames, the latter shows similar circumscribed ellipses.

6.3 Gridshell structures

Paneling with planar faces and torsion-free support structures. A principal curvature net is an orthogonal conjugate net, corresponding to an orthogonal planar quad mesh in the discrete case. A key property is that the envelopes of surface normals along principal curvature lines are developable, with the two families intersecting along straight lines—the surface normals themselves. Such nodes, where two developable strips meet along a straight line segment (a ruling of both strips), are called *torsion-free* (see inset). Torsion-free nodes facilitate fabrication and multi-layer architectural paneling [Liu et al. 2006]: faces can be covered by nearly rectangular planar panels, while the underlying support structure (normal developables) remains developable and orthogonal to the surface.

We design a principal curvature net via an orthogonal DIS_{diag} -net with planar faces, rendered as an architectural structure with torsion-free supports, decorated with planar glass and similar parallelograms. The tracing edges of the parallelograms form two families of discrete curves with constant angles, creating a regular pattern that casts interesting shadow patterns under sunlight (Fig. 30).



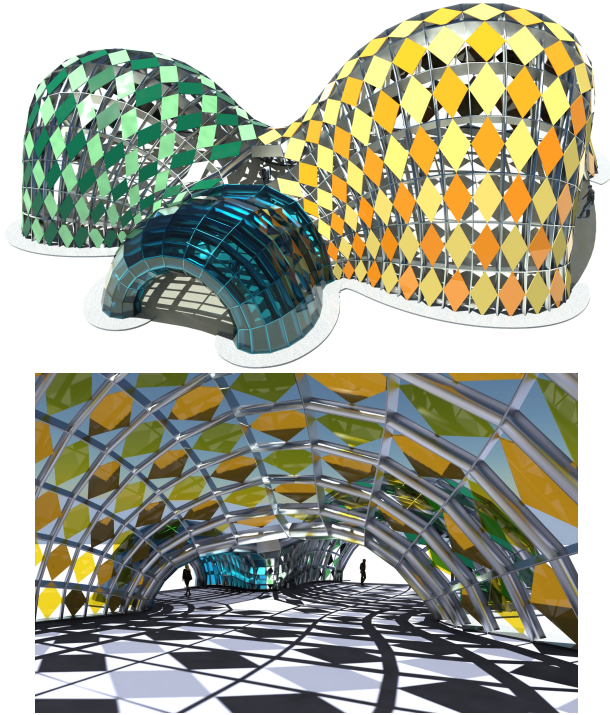


Fig. 30. A DIS_{diag} -net with planar faces yields a principal curvature net. The steel gridshell possesses torsion-free nodes and three types of planar panels: near-rectangular glass panels, similar $(t, \bar{t})$ -parallelograms (green) at $t = 0.4$, and similar mid-edge rhombi (yellow) at $t = 0.5$. These form a regular pattern with constant-angle tracing edges that produce distinct shadow patterns under sunlight.

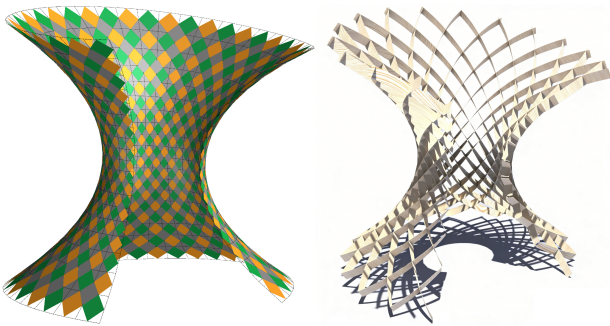


Fig. 31. A minimal gridshell structure achieved by a DIS_{diag} -net optimized as an orthogonal A-net in the diagonal direction. Left: CBP with planar quad faces. Right: Bended wooden lamellae as straight developable strips orthogonal to the asymptotic (diagonal) curves form an architectural gridshell with orthogonal intersections.

Asymptotic gridshell structure. Two families of intersecting developable strips orthogonal to the underlying surface can be achieved not only by a principal curvature net but also by an *asymptotic net*, where the nodes are relaxed (not torsion-free), i.e., the strips meet along curved lines rather than straight surface normals. These developable strips correspond to flexible lamellae with bending behavior, forming asymptotic gridshell structures over negatively curved shapes. Such structures are cost-effective due to repetitive

elements—straight lamellae and constant-angle nodes—as demonstrated in recent built gridshells [Jiang et al. 2021b; Pellis et al. 2020a; Schling et al. 2018, 2023].

Geometrically, an asymptotic net with constant intersection angle characterizes surfaces with constant ratio of principal curvatures [Wang and Pottmann 2022]; the orthogonal case degenerates to minimal surfaces. A discrete asymptotic net is characterized by planar vertex stars everywhere. Thus, starting from a DIS_{diag} -net, enforcing planarity of vertex stars in the diagonal mesh together with right angles between diagonals produces a planar CBP covering a minimal surface (Fig. 31).

6.4 Paper cutting models

A planar sheet can be deformed into curved 3D shapes through various methods: pure bending [Rabinovich 2020; Signer et al. 2021]; marking curves and folding along them [Jiang et al. 2019a; Kilian et al. 2008; Rabinovich 2020; Tang et al. 2016]; or cutting seams and stretching to open them [Jiang et al. 2020a, 2022; Luković 2019; Segall et al. 2025, 2026; Zaman et al. 2025]. Complex freeform surfaces can be approximated by piecewise paper patches with well-aligned, glueable boundaries [Inza et al. 2023; Jiang et al. 2020b; Rabinovich 2020; Stein et al. 2018; Zhao et al. 2023].

The proposed DIC_{diag} -net facilitates shape transformations through paper-based fabrication. As the mesh is locally isometric to a rectangular grid, its edges naturally align with cutting features. We demonstrate two interactive paper-cutting designs with prescribed seams to construct lightweight 3D models (Fig. 32).

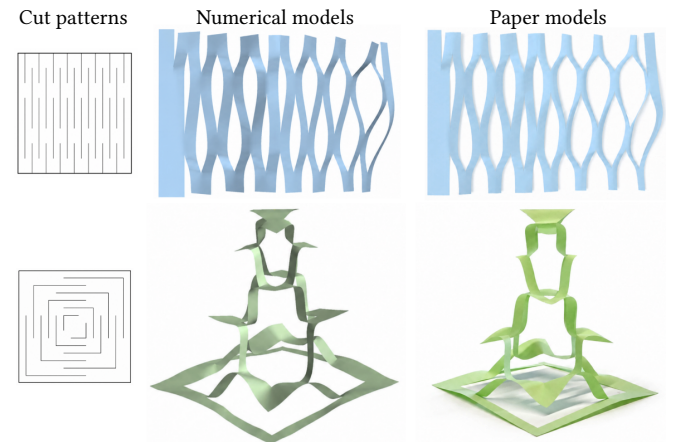


Fig. 32. 3D paper-cutting models with complex topology achieved by interactive editing of planar rectangular patches with designed seams; numerical models align well with physical results. Top: left strip fixed, right strip pulled outward. Bottom: center pulled upward, four corner squares fixed.

Limitations. Although DIC_{diag} -nets discretize developable surfaces independently of rulings, they remain tied to parametrizations locally isometric to rectangular regions. Patterns may change under affine maps and offsets, leaving limited degrees of freedom—particularly for complex CAD surface approximation by piecewise developables, freeform origami, or kirigami designs—which we leave for future research.

6.5 Geometry constraints and optimization

Based on the CBP method, the DI_{diag} -nets, DIS_{diag} -nets, and DIC_{diag} -nets characterized and formulated in Tab. 1 are well aligned with the guided projection algorithm [Tang et al. 2014], a Gauss-Newton method that involves the optimization of at most quadratic constraints. The numerical optimization converges efficiently for almost any reasonable input quad mesh.

Constraint for gliding parallelograms. The energy of the gliding edge point $\mathbf{p}_i$ together with edge ratio t from Eq. (2) can be written as below

$$E_{\text{CBP}} = \sum_{(i,j) \in \text{half-edges}} \|t\mathbf{v}_i + (1-t)\mathbf{v}_j - \mathbf{p}_i\|^2.$$

Ratio t is set to between 0 and 1. If $t = 0$ or $t = 1$, then the gliding point $\mathbf{p}_i$ agree with one of the edge endpoint that is a vertex. Such range constraint can be represented by two auxiliary variables ϵ_1 and ϵ_2 that is $t = \epsilon_1^2$, $1 - t = \epsilon_2^2$. The energy term is

$$E_t = (t - \epsilon_1^2)^2 + (t + \epsilon_2^2 - 1)^2.$$

Constraint for DI_{diag} -net. Eq. (3) is enforced via the energy

$$E_{\text{DI}} = \sum_{f \in \text{quads}} [(\mathbf{v}_3^f - \mathbf{v}_1^f) \cdot (\mathbf{v}_4^f - \mathbf{v}_2^f) - c_0 \|\mathbf{v}_3^f - \mathbf{v}_1^f\| \|\mathbf{v}_4^f - \mathbf{v}_2^f\|]^2,$$

where $c_0 = \cos \theta$ is a global variable and the products $\|\mathbf{v}_3^f - \mathbf{v}_1^f\| \cdot \|\mathbf{v}_4^f - \mathbf{v}_2^f\|$ can be taken from the previous iteration. Alternatively, introducing auxiliary variables $l_1^f, l_2^f, \mathbf{u}_1^f, \mathbf{u}_2^f$ satisfying Eqs. (4), the energy becomes

$$E_{\text{DI}} = \sum_{f \in \text{quads}} \left\{ (\mathbf{u}_1^f \cdot \mathbf{u}_2^f - c_0)^2 + [\|\mathbf{u}_1^f\|^2 - 1]^2 + [\|\mathbf{u}_2^f\|^2 - 1]^2 + \|\mathbf{v}_3^f - \mathbf{v}_1^f - l_1^f \mathbf{u}_1^f\|^2 + \|\mathbf{v}_4^f - \mathbf{v}_2^f - l_2^f \mathbf{u}_2^f\|^2 \right\}$$

Constraint for DIS_{diag} -net. Eq. (5) is enforced jointly with the DI energy E_{DI} by the single penalty term

$$E_{\text{DIS}} = E_{\text{DI}} + \sum_{t \in \square} (l_2^t - l_1^t r_0)^2 + \sum_{i \in \triangle} (l_1^i - l_2^i r_0)^2,$$

where r_0 is an auxiliary variable that converges to 1, and the edge-length variables $l_1^t, l_2^t, l_1^i, l_2^i$ are reused from E_{DI} .

Additionally, the similarity constraint can be formulated either on E_{CBP} or directly in terms of λ and t :

$$E_{\text{DIS}} = E_{\text{DI}} + E_t + \sum_{f \in \text{quads}} (at + t - 1)^2 + \sum_{t \in \square} (\lambda l_1^t - at_2^t)^2 + \sum_{i \in \triangle} (\lambda l_2^i - at_1^i)^2,$$

where a defined as $\frac{1-t}{t}$ is an auxiliary variable that lowers the equation order. A user-specified target ratio $\bar{\lambda}$ is imposed via the penalty $E_{\bar{\lambda}} = (\lambda - \bar{\lambda})^2$.

Constraint for DIC_{diag} -net. The energy corresponding to Eq. (6) can either exploit the edge-length variables l_1^f, l_2^f already present in E_{DI} to enforce constancy, or be expressed directly through three

scalars c_1, c_2, c_3 :

$$E_{\text{DIC}} = \sum_{t \in \square} \left\{ [\|\mathbf{v}_3^t - \mathbf{v}_1^t\|^2 - c_1^2]^2 + [\|\mathbf{v}_4^t - \mathbf{v}_2^t\|^2 - c_2^2]^2 \right\} + \sum_{i \in \triangle} \left\{ [\|\mathbf{v}_3^i - \mathbf{v}_1^i\|^2 - c_2^2]^2 + [\|\mathbf{v}_4^i - \mathbf{v}_2^i\|^2 - c_1^2]^2 \right\} + \sum_{f \in \text{quads}} [(\mathbf{v}_3^f - \mathbf{v}_1^f) \cdot (\mathbf{v}_4^f - \mathbf{v}_2^f) - c_3]^2,$$

where c_1, c_2, c_3 fix the shape of the congruent parallelograms. The constant interior angle θ of the diagonal mesh is driven by a user target $\bar{\theta}$ via Eq. (7):

$$E_{\theta} = (\cos \bar{\theta} c_1 c_2 - c_3)^2.$$

Fairness constraint. Visual polyline artefacts are controlled by the fairness of every triple of consecutive points $\mathbf{p}_{il}, \mathbf{p}_i, \mathbf{p}_{ir}$, whether they lie on the control mesh or on the diagonal mesh:

$$E_{\text{Vfair}} = \sum_{i \in \text{polyline}} (2\mathbf{p}_i - \mathbf{p}_{il} - \mathbf{p}_{ir})^2.$$

The curve-like quality of the Gauss image of a DIC_{diag} -net is influenced by the vanishing oriented area controlled by Eq. (9) for each quad's four vertex normals (Eq. (8)):

$$E_{\text{Nfair}} = \sum_{k \in \text{vertices}} \left\{ ((\|\mathbf{n}^k\|^2 - 1)^2 + [\mathbf{n}^k \cdot (\mathbf{v}_3^k - \mathbf{v}_1^k)]^2 + [\mathbf{n}^k \cdot (\mathbf{v}_4^k - \mathbf{v}_2^k)]^2) \right\} + \sum_{f \in \text{quads}} \|(\mathbf{n}_3^f - \mathbf{n}_1^f) \times (\mathbf{n}_4^f - \mathbf{n}_2^f)\|^2.$$

Target function. The overall optimization objective integrates the above energies with adjustable weights:

$$E = E_{\square} + E_{*} + \omega_{\theta} E_{\theta} + \omega_{\text{Vf}} E_{\text{Vfair}} + \omega_{\text{Nf}} E_{\text{Nfair}},$$

where $E_{\square}$ denotes DI_{diag} -net, DIS_{diag} -net, or DIC_{diag} -net energy; E_{*} is optional and application-dependent, encoding additional requirements such as planar quad faces (principal curvature lines) or planar vertex stars (asymptotic nets) [Jiang et al. 2019b, 2020b, 2021b]; $\omega_{\theta} \in \{0, 1\}$ selects prescribed constant angle; $\omega_{\text{Vf}} \in [0, 0.01]$ controls vertex fairness; and $\omega_{\text{Nf}} \in [0.01, 1]$ is activated for DIC_{diag} -nets to enforce normal fairness. Other user-specified constraints—surface approximation, boundary vertices gliding on curves or planes—follow [Dellinger et al. 2023] and [Wang et al. 2025] and are added with weights below 1.

Statistics. Our optimization algorithm is implemented in Python on a 2.7 GHz Intel® Core i5 CPU with 8GB RAM. Computational statistics for representative examples (Fig. 15) are listed in Tab. 2.

Table 2. Optimization statistics for the DI_{diag} -net, DIS_{diag} -net, and DIC_{diag} -net shown in Fig. 15. The gradual optimization process converges within at most 25 iterations. Fairness weights vary by stage: $\omega_{\text{Vf}} = 0.0005$ for iterations 1–10 (c_1, c_2) or 1–15 (c_3); $\omega_{\text{Nf}} = 0.1$ for iterations 11–20; $\omega_{\text{Vf}} = \omega_{\text{Nf}} = 0$ for final 5 iterations.

Fig. 15.	$ F $	#var	#cons	residual	#iter	time[s]/it
(c ₁)	400	4524	3602	$1.34e - 12$	15	4.32
(c ₂)	400	4525	4002	$4.19e - 13$	15	8.94
(c ₃)	400	1326	1220	$1.78e - 12$	25	1.69

7 CONCLUSION

Our paper initiates discussion of isogonal nets for a richer class of surfaces by controlling angles, similarity, and congruence of edge-gliding parallelograms via the CBP method on quad meshes. We present a framework grounded in discrete differential geometry, motivated by architectural applications but expected to have broader impact. Computational design studies and physical prototypes illustrate a wide range of possible applications.

Future work. Quad patterns with similar equivalence sub-panels offer a new class of force-, function-, and fabrication-aware designs with applications across many domains: large-scale architectural facades and shell structures, auxetic metamaterials, periodic microstructures, clothing, and industrial products. For instance, the similarity property of inscribed parallelograms enables discrete equivalence for linkage quad panels functioning as auxetic panels. Their inverse design on rationalized surfaces involves conformal geometry with compatible discrete curvature notions and fast programmable tools. Shape deformation behavior—shrinkage and expansion limits—depends on material physics, careful design of contacting structures, and gravitational loading during deployment.

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APPENDIX

A.1 Geodesic curvatures of a Chebyshev net.

Proposition. For a Chebyshev net with angle $\theta(u, v)$ the geodesic curvatures of the u - and v -curves are

$$\kappa_g^u = -\theta_u, \quad \kappa_g^v = -\theta_v.$$

PROOF. Use the natural parametrization Eq. (1): $E = G = 1$, $F = \cos \theta$. For a u -curve the unit tangent is $\mathbf{T} = \mathbf{X}_u$; its geodesic curvature is

$$\kappa_g^u = \mathbf{T}' \cdot (\mathbf{N} \times \mathbf{T}) = \mathbf{X}_{uu} \cdot \frac{\mathbf{X}_v - \cos \theta \mathbf{X}_u}{\sin \theta},$$

where $\mathbf{N} = \mathbf{X}_u \times \mathbf{X}_v / \sin \theta$ is the surface normal. By the Gauss formula

$$\mathbf{X}_{uu} = \Gamma_{uu}^u \mathbf{X}_u + \Gamma_{uu}^v \mathbf{X}_v + \mathbf{L}\mathbf{N}.$$

Orthogonality of $\mathbf{X}_u, \mathbf{X}_v$ to $\mathbf{N}$ and $\mathbf{X}_{uu} \cdot \mathbf{X}_u = 0$ give

$$\kappa_g^u = \frac{1}{\sin \theta} (\Gamma_{uu}^v \sin^2 \theta) = \Gamma_{uu}^v \sin \theta.$$

The Christoffel symbol is

$$\Gamma_{uu}^v = \frac{2F_u - FE_u - EF_u}{2(EG - F^2)} = \frac{-\theta_u \sin \theta}{\sin^2 \theta} = -\frac{\theta_u}{\sin \theta}.$$

Hence $\kappa_g^u = -\theta_u$. Interchanging $u \leftrightarrow v$ yields $\kappa_g^v = -\theta_v$. $\square$

A.2 Ellipses construction.

Similar parallelograms of a DIS_{diag} -net have similar area-maximizing inscribed ellipses and area-minimizing circumscribed ellipses (Prop. 4.1), constructed as follows.

Inscribed ellipses with max areas. The area-maximizing ellipse inscribed in a parallelogram is tangent to the edge midpoints. Within similar parallelograms, these ellipses are similar and can be constructed via affine transformations [Ma and Zhou 2024].

Circumscribed ellipses with minimal areas. The major axes of area-minimizing circumscribed ellipses align with the long diagonals of the parallelograms. Consider a parallelogram with vertices

$\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4$ and maximal diagonal $\mathbf{p}_1\mathbf{p}_3$. The endpoints $\mathbf{p}_1$ and $\mathbf{p}_3$ define the ellipse's major axis, giving semi-major axis length $a = \frac{\|\mathbf{p}_1\mathbf{p}_3\|}{2}$. Points $\mathbf{p}_2$ and $\mathbf{p}_4$ are symmetric about the barycenter O . Positioning O at the origin with $\mathbf{p}_1, \mathbf{p}_3$ on the x -axis, let $d = \|\mathbf{p}_2O\|$. The ellipse equation is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with semi-minor axis length b . Since $d^2 = x^2 + y^2$, substituting $y^2 = b^2(1 - \frac{x^2}{a^2})$ yields

$$d^2 = x^2 + \left(1 - \frac{x^2}{a^2}\right)b^2.$$

Solving for b gives

$$b = a\sqrt{\frac{d^2 - x^2}{a^2 - x^2}},$$

which uniquely determines the circumscribed ellipse.

