May 28, 2025

Discrete Isogonal Nets with Similar Parallelograms

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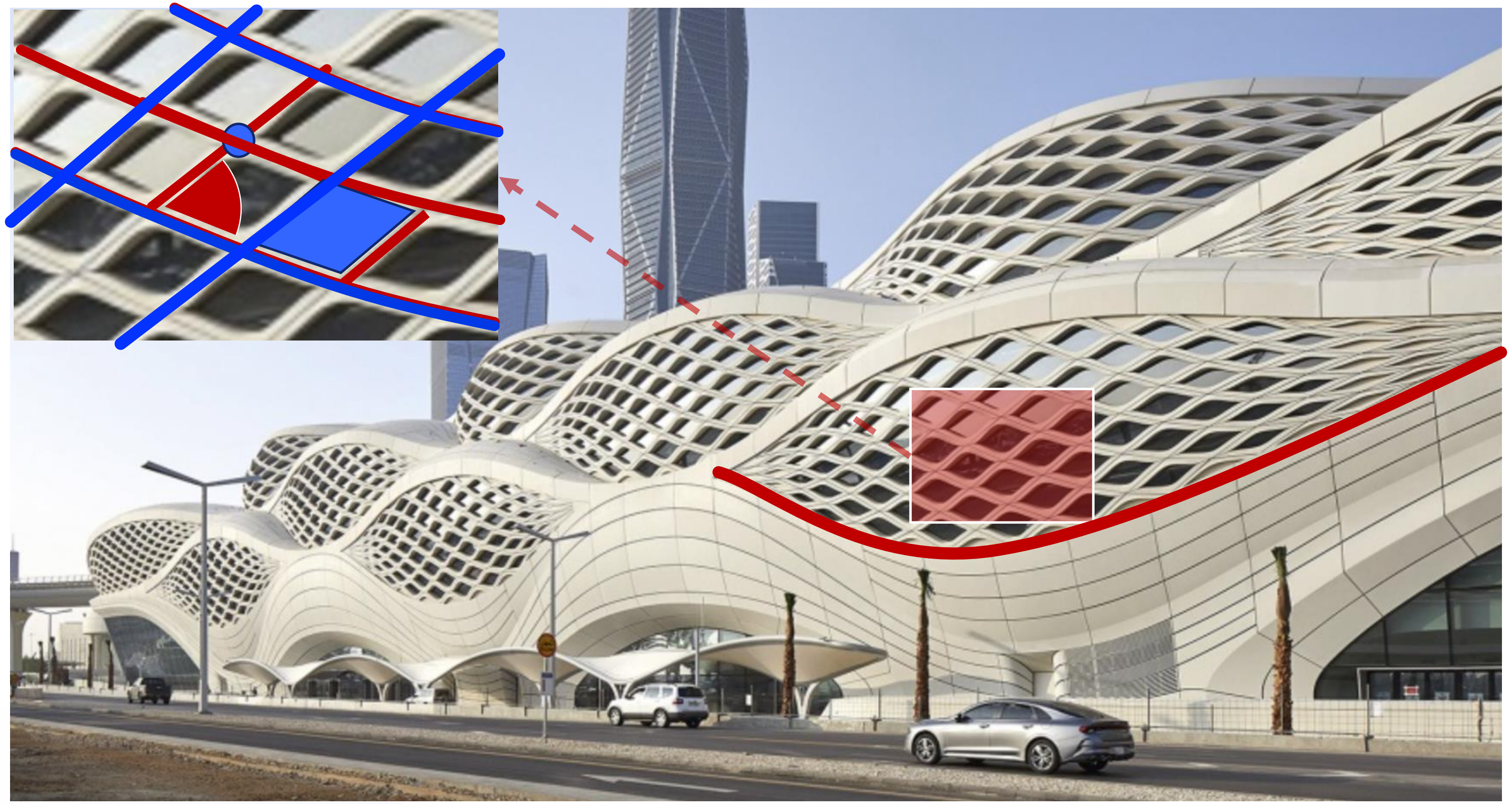
Background

- High-quality surfaces are essential in architecture and product design



KAFD Metro Station in Riyadh, designed by ZHA

- regular vertex
- smooth curve
- equal opposite sides
- right angle
- flat face
- tessellation density
- feature line alignment

Background

Orthogonal nets (O-nets)

- basic for other specialized quad nets,
  e.g. principal curvature net, minimal net,…
- offer aesthetic appeal and engineering practicality

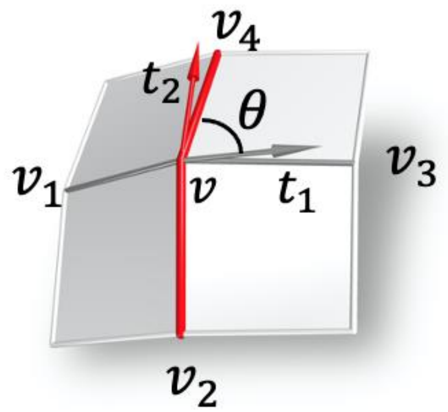
Yas Hotel, Abu Dhabi

Isogonal nets (I-nets)

- Constant curve angle beyond 90°
- Compelling alternative to O-nets
- Broader control over shape
- Unique visual effects
- Adaptable structural layouts

Phoenix Media Center, Beijing

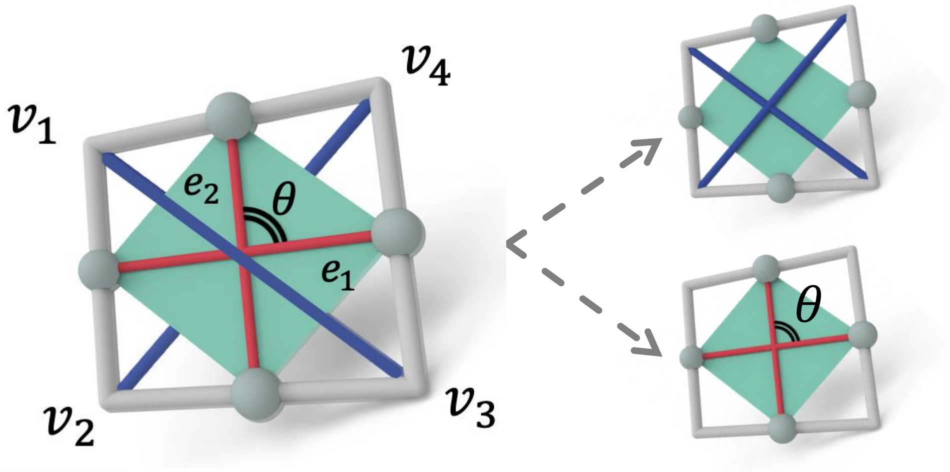
Overview of I-nets



O-net: $\theta = \frac{\pi}{2} \Leftrightarrow (e_3 - e_1)\cdot(e_4 - e_2) = 0$ ⇠ $e_i = \frac{v_i - v}{||v_i - v||}, (i = 1,2,3,4)$

I-net: $\theta = const. \Leftrightarrow\ t_1 \cdot t_2 = \cos\theta$ ⇠ $t_1 = \frac{e_3 - e_1}{||e_3 - e_1||}, t_2 = \frac{e_4 - e_2}{||e_4 - e_2||}$

Defined unit tangents - - - -> increase computation cost



Mid-edge subdivided quad is a parallelogram ▱

O-net: $\theta = \frac{\pi}{2} \Leftrightarrow (v_3 - v_1)^2 = (v_4 - v_2)^2$ ◊

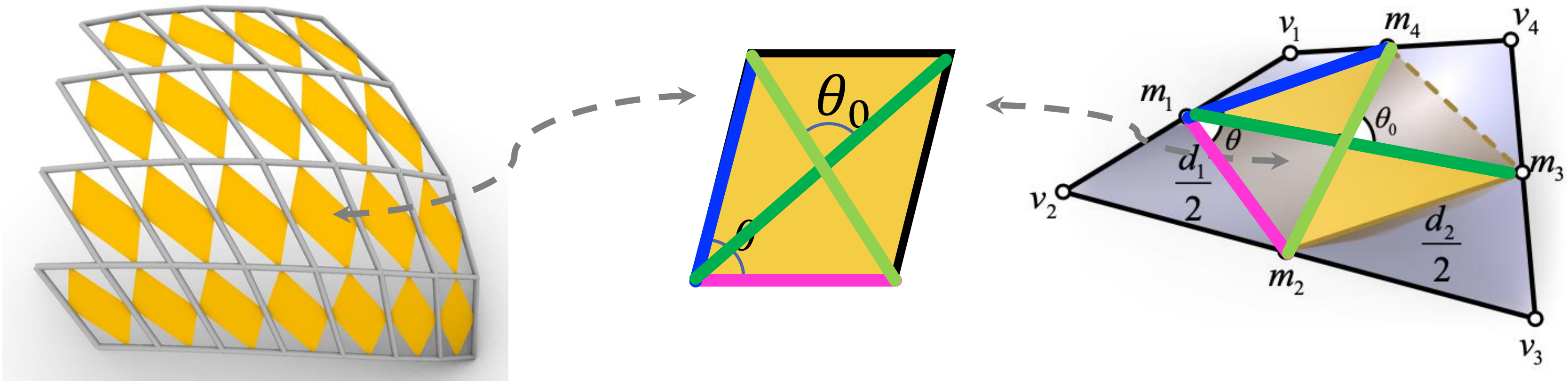
I-net: $\theta = const. \Leftrightarrow$?

Checkerboard pattern (CBP) - - - - - - -> How to represent I-net by edge lengths?

Felix Dellinger, Xinye Li, Hui Wang#. *Discrete orthogonal structures*. Computers & Graphics 2023.

Proposed Method

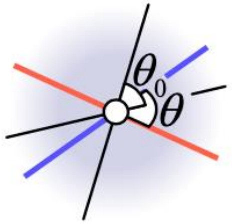
- I-nets using similar mid-edge subdivided parallelograms



$$\frac{||d_2||}{||d_1||} = \lambda$$

$$\frac{||m_1 m_3||}{||m_2 m_4||} = \mu$$

- Constraining edge ratios instead of angles
- Generalizations of O-nets
  - Rhombus
  - Rectangle
  - Square



- Generalization to isogonal 4-webs (I-webs)
- Versatility of I-nets and I-webs in high-quality surface

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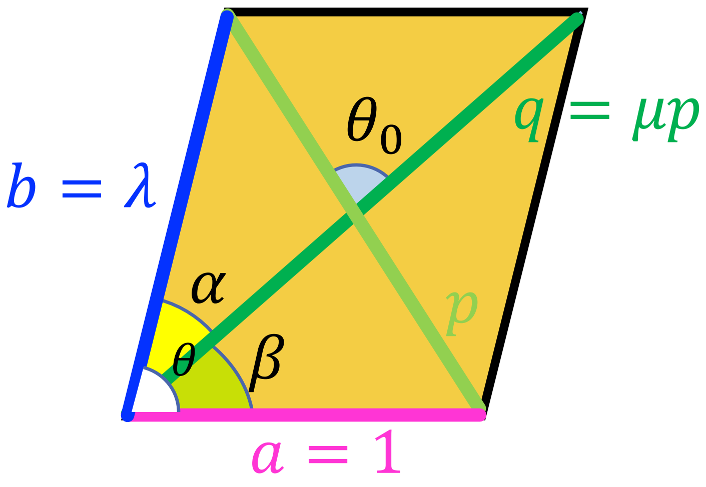
Parallelogram Geometry

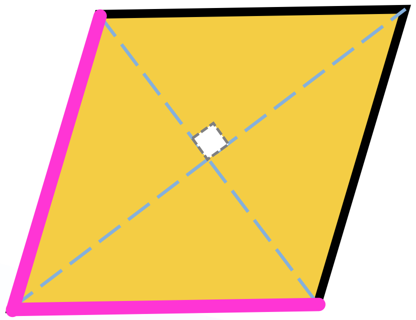


$$a:b:p:q = 1:\lambda:\sqrt{\frac{2(1+\lambda^2)}{1+\mu^2}}:\mu\sqrt{\frac{2(1+\lambda^2)}{1+\mu^2}}$$

$$\cos\theta = \frac{(1+\lambda^2)(\mu^2-1)}{2\lambda(1+\mu^2)}, \cos\theta_0 = \frac{(1+\mu^2)(\lambda^2-1)}{2\mu(1+\lambda^2)}$$

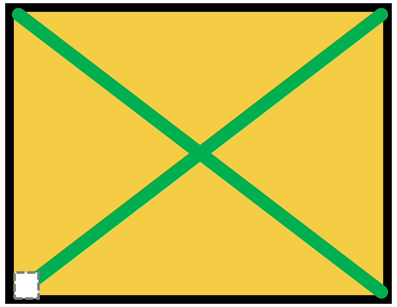
$$\cos\alpha = \frac{3\lambda^2\mu^2+\lambda^2+\mu^2-1}{2\lambda\mu\sqrt{2(1+\lambda^2)(1+\mu^2)}}, \cos\beta = \frac{\lambda^2\mu^2-\lambda^2+3\mu^2+1}{2\mu\sqrt{2(1+\lambda^2)(1+\mu^2)}}$$

$\lambda = 1 \Leftrightarrow \theta_0 = \frac{\pi}{2}$



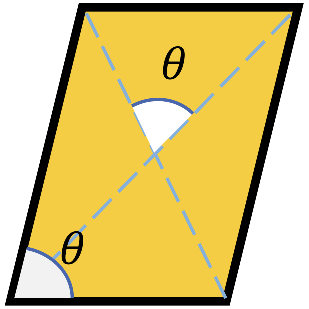
Rhombus

$\mu = 1 \Leftrightarrow \theta = \frac{\pi}{2}$



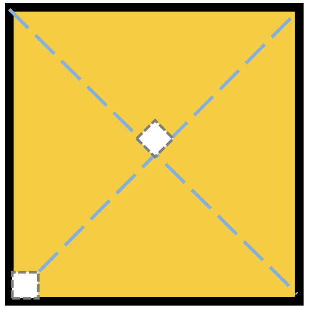
Rectangle

$\lambda = \mu \Leftrightarrow \theta = \theta_0$



θ-Parallelogram

$\lambda = \mu = 1 \Leftrightarrow \theta = \theta_0 = \frac{\pi}{2}$



Square

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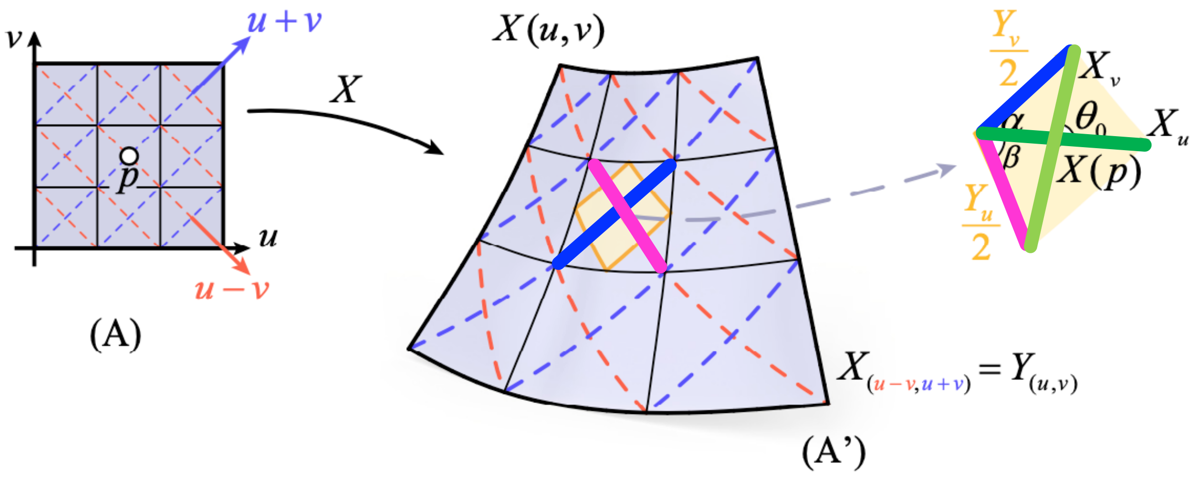
Isogonal Nets and Webs

Surface parametrization:

$$X: (u,v) \subset \mathbb{R}^2 \rightarrow X(u,v) \subset \mathbb{R}^3$$

Diagonal parametrization:

$$Y(u,v) = X(u-v, u+v)$$



constant $\frac{||X_u||}{||X_v||} := \mu$

constant $\frac{||Y_v||}{||Y_u||} := \lambda$

$\Longrightarrow$

$$Y_v^2 = \lambda^2\, Y_u^2 \Leftrightarrow (X_u + X_v)^2 = \lambda^2\, (X_u - X_v)^2$$

$$\Downarrow$$

$$\theta(\lambda,\mu), \theta_0(\lambda,\mu), \alpha(\lambda,\mu), \beta(\lambda,\mu)$$

$$\cos\theta = \frac{(1+\lambda^2)(\mu^2-1)}{2\lambda(1+\mu^2)},$$

$$\cos\theta_0 = \frac{(1+\mu^2)(\lambda^2-1)}{2\mu(1+\lambda^2)}$$

$$\cos\alpha = \frac{3\lambda^2\mu^2+\lambda^2+\mu^2-1}{2\lambda\mu\sqrt{2(1+\lambda^2)(1+\mu^2)}},$$

$$\cos\beta = \frac{\lambda^2\mu^2-\lambda^2+3\mu^2+1}{2\mu\sqrt{2(1+\lambda^2)(1+\mu^2)}}$$

Theorem

If the ratios $\frac{||X_u||}{||X_v||}$ and $\frac{||Y_v||}{||Y_u||}$ are constant, $X(u,v)$ is **isogonal (I-net)**, and $X(u,v)$ and $Y(u,v)$ form an **isogonal 4-web (I-web)**.

Discrete Isogonal Nets and Webs

Quad mesh $f: \mathbb{Z}^2 \to \mathbb{R}^3$:

vertices $v_i(1, \dots, 4)$ in each quad face

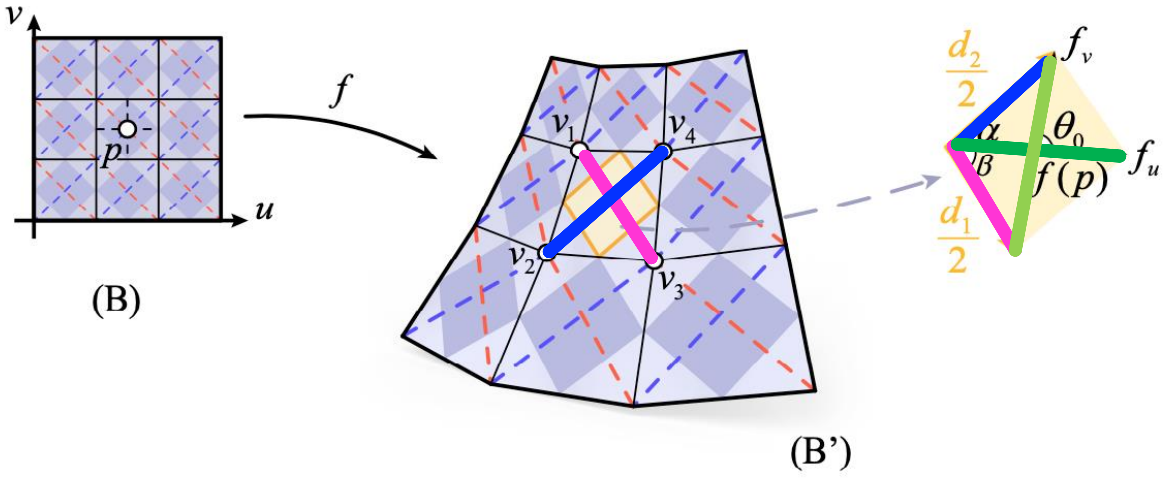
Discrete first-order smooth derivatives :

$$d_1 = v_3 - v_1, \; d_2 = v_4 - v_2$$

Discrete second-order smooth derivatives :

$$f_u = \frac{1}{2}(v_3 + v_4 - v_1 - v_2), \; f_v = \frac{1}{2}(v_1 + v_4 - v_2 - v_3)$$

constants $\frac{||d_2||}{||d_1||} := \lambda, \frac{||f_u||}{||f_v||} := \mu \quad \Longrightarrow \quad \theta(\lambda,\mu), \theta_0(\lambda,\mu), \alpha(\lambda,\mu), \beta(\lambda,\mu)$



Theorem

If the diagonal ratio $\frac{||d_2||}{||d_1||}$ and medial-line ratio $\frac{||f_u||}{||f_v||}$ are constant, f is **isogonal (DI-net)**, and f and its diagonal quad mesh d form a discrete **isogonal 4-web (DI-web)**.

Special cases

- $\lambda = 1 \iff \theta_0 = \frac{\pi}{2} \iff X_u \perp X_v$

  Definition. An I-net is an orthogonal net (**O-net**) if λ=1.

  Definition. A DI-net is orthogonal (**DO-net**) if λ=1.

  Corollary. A DO-net consists of similar rhombuses in the CBP.

- $\mu = 1 \iff \theta = \frac{\pi}{2} \iff Y_u \perp Y_v$

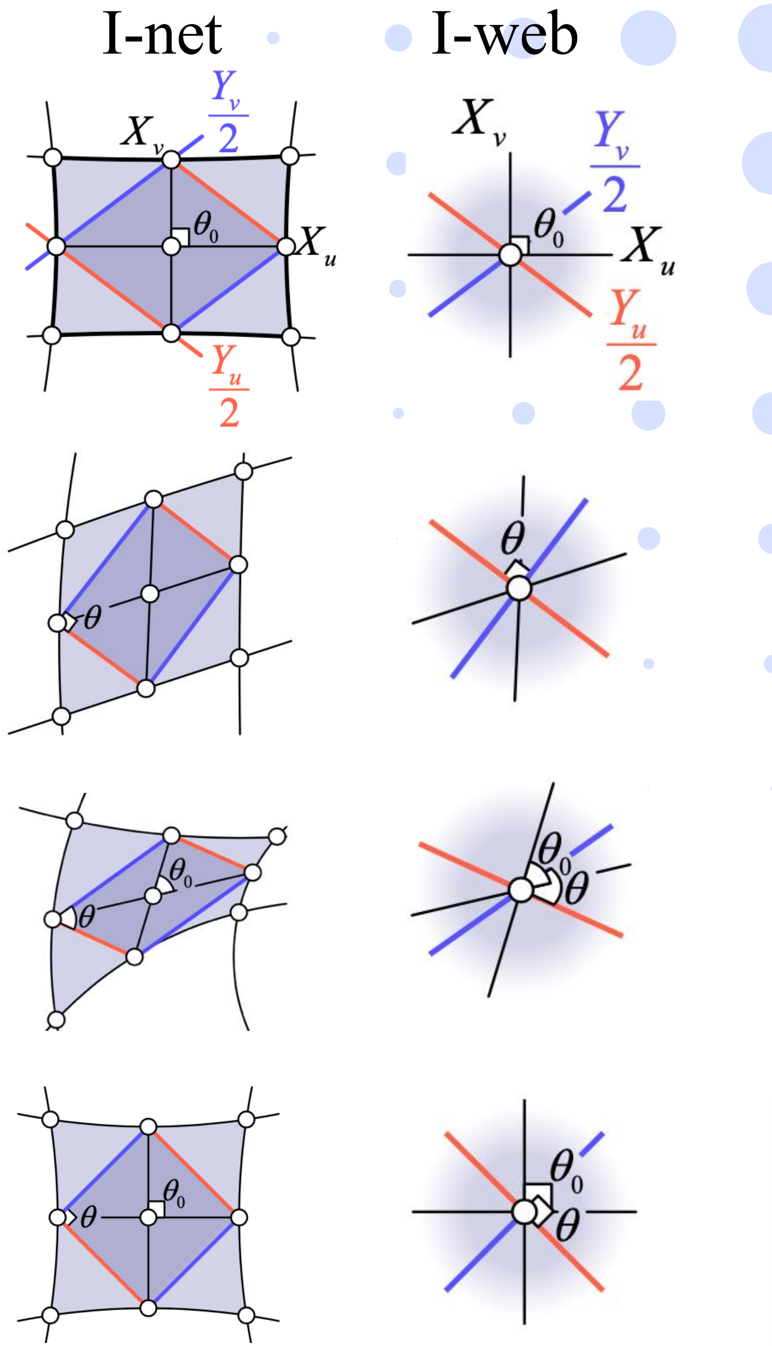
  Corollary. A DI-net with μ=1 consists of similar rectangles in the CBP.

- $\lambda = \mu \iff \theta = \theta_0 \iff \frac{||X_u||}{||X_u||} = \frac{||X_u+X_v||}{||X_u-X_v||}$

  Corollary. A DI-net with θ-Quads consists of similar θ-Parallelograms in the CBP.

- $\lambda = \mu = 1 \iff \theta = \theta_0 = \frac{\pi}{2} \iff X_u \perp X_v,\ Y_u \perp Y_v$

  Corollary. A DO-net with θ-Quads consists of squares in the CBP.

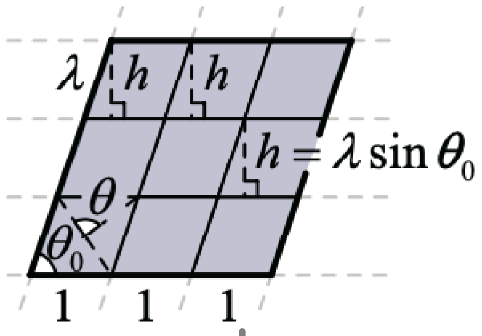


Construction of I-nets

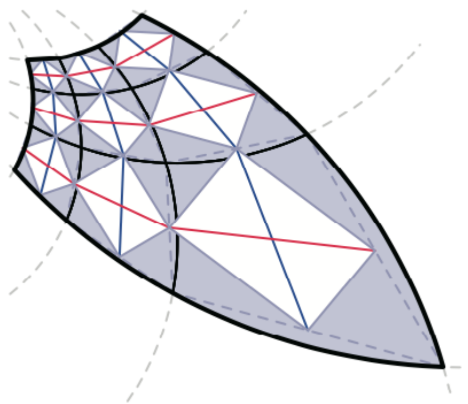
- I-nets on planar sheared grid
- I-nets on developable surfaces
- I-nets from isothermal parametrizations
- I-nets by computational design

Construction of I-nets

- I-nets on planar sheared grid



$\omega = \frac{1}{z}$



$$C_1 = \{z = t + i\mathcal{A}, \frac{\mathcal{A}}{h} \subseteq \mathbb{Z}, 0 \notin \mathcal{A}, h \in \mathbb{R}\}$$

$$C_2 = \{z = t + \mathcal{B} + i(kt), k = \tan\theta \in \mathbb{R}, \mathcal{B} \subseteq \mathbb{Z}, 0 \notin \mathcal{B}\}$$

$\omega = \frac{1}{z}$

$$u+iv = \frac{t}{t^2+\mathcal{A}^2} - i\frac{\mathcal{A}}{t^2+\mathcal{A}^2}$$

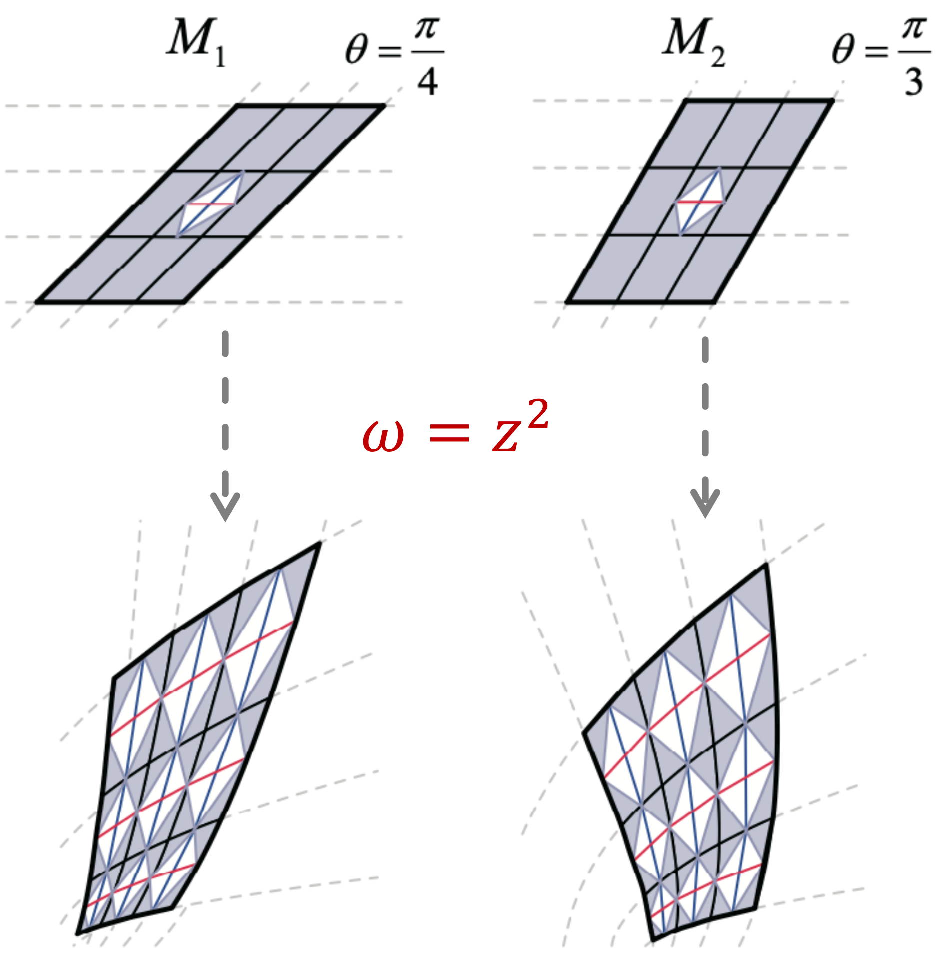
$$u + iv = \frac{t+\mathcal{B}}{(1+k^2)t^2+2\mathcal{B}t+\mathcal{B}^2} - i\frac{kt}{(1+k^2)t^2+2\mathcal{B}t+\mathcal{B}^2}$$

$$\Downarrow$$

$$u^2 + (v + \frac{1}{2\mathcal{A}})^2 = \frac{1}{4\mathcal{A}^2}, \quad (u - \frac{1}{2\mathcal{B}})^2 + (v - \frac{1}{2k\mathcal{B}})^2 = \frac{1+k^2}{4k^2\mathcal{B}^2}$$

Construction of I-nets

- I-nets on planar sheared grid



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$$C_2 = \{z = t + \mathcal{B} + i(kt), k = \tan\theta \in \mathbb{R}, \mathcal{B} \subseteq \mathbb{Z}, 0 \notin \mathcal{B}\}$$

$\omega = z^2$

$$u + iv = t^2 - \mathcal{A}^2 + i(2\mathcal{A}t)$$

$$u + iv = (1 - k^2)t^2 + 2\mathcal{B}t + \mathcal{B}^2 + i(2kt^2 + 2k\mathcal{B}t)$$

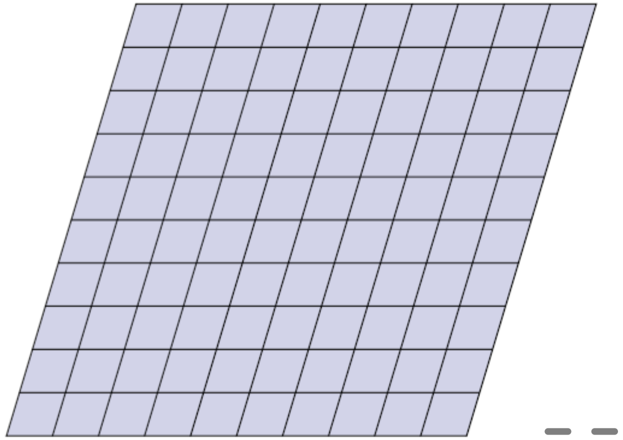
$\Downarrow$

$$v^2 - 4\mathcal{A}^2u - 4\mathcal{A}^4 = 0$$

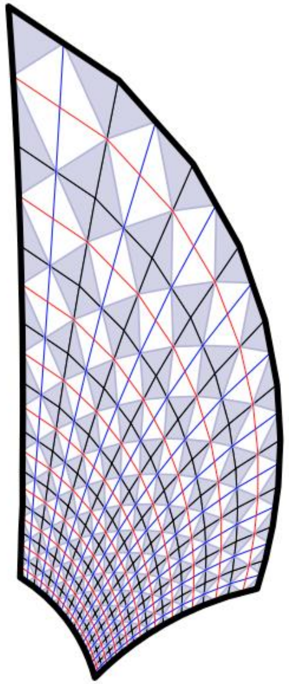
$$4k^2u^2 + (k^2 - 1)^2v^2 + 4k(k^2 - 1)uv + 4k^2\mathcal{B}^2(k^2 - 1)u - 8k^3\mathcal{B}^2v - 4k^4\mathcal{B}^4 = 0.$$

Construction of I-nets

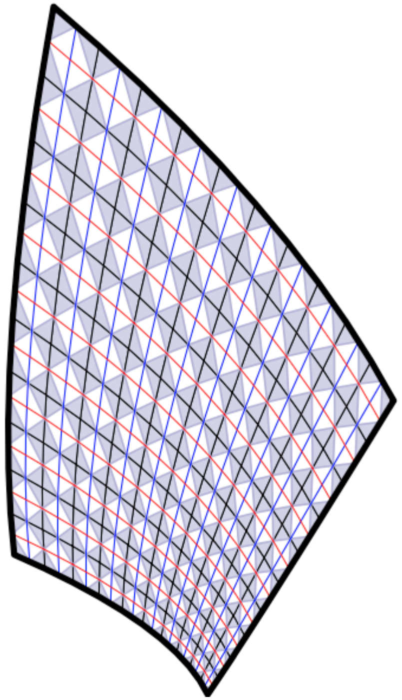
- I-nets on planar sheared grid



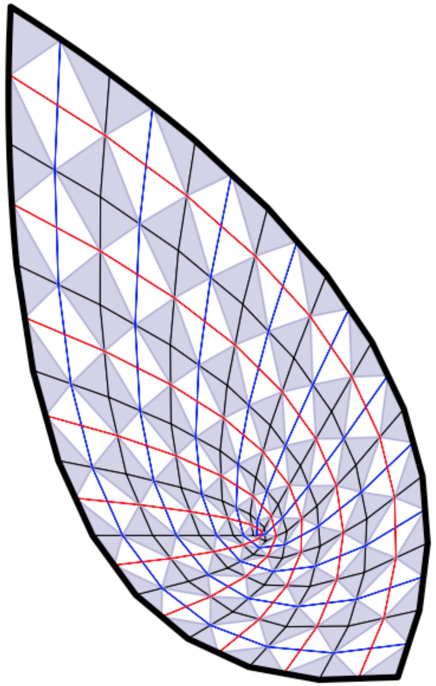
10 x 10 grid



$\omega = \dfrac{1}{z}$



$\omega = z^2$

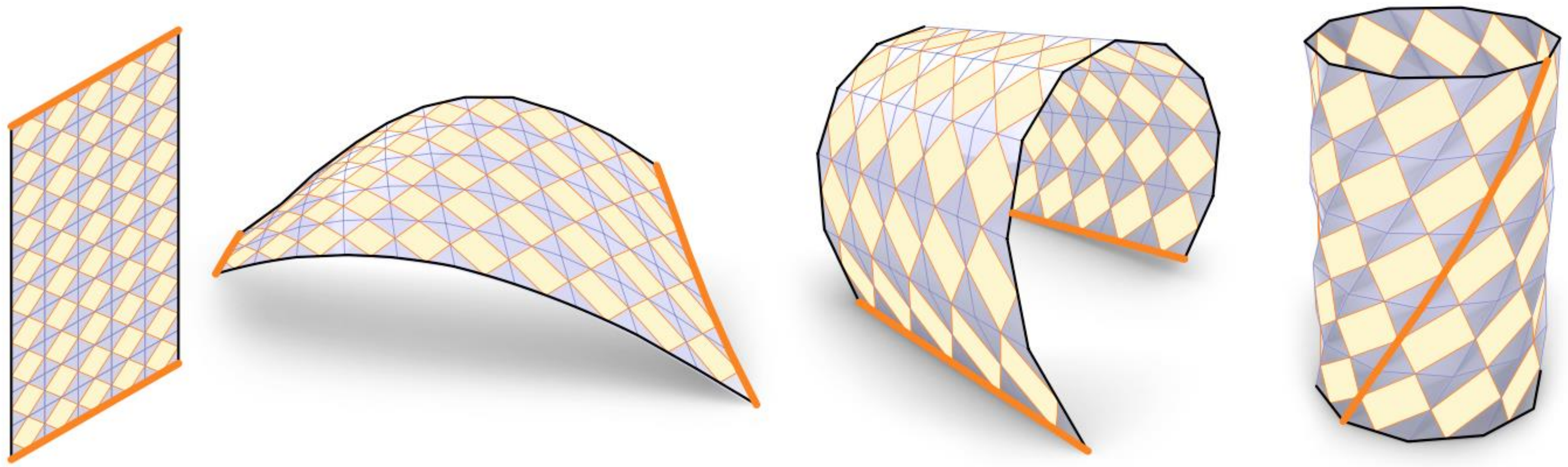


$\omega = z^2$

(Origin lies within the pre-image)

Construction of I-nets

- I-nets on developable surfaces



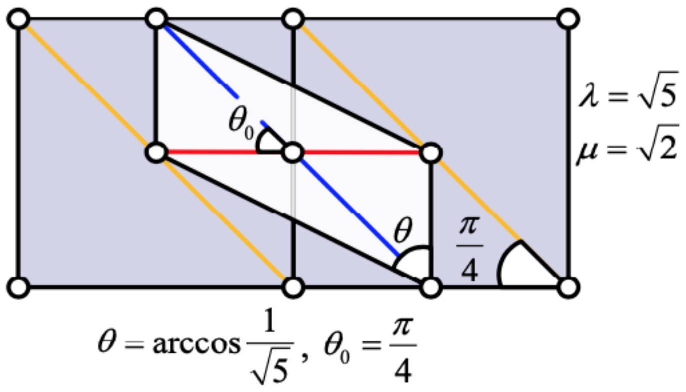
developable I-nets through isometric deformations

Construction of I-nets

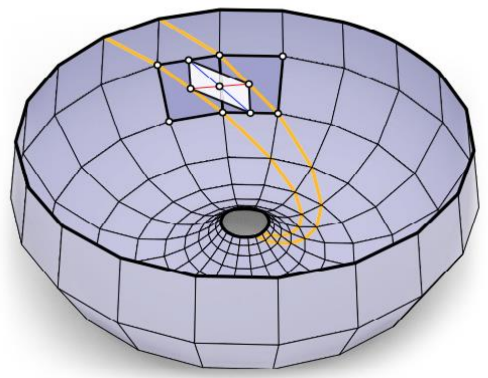
- I-nets from isothermal parametrizations

Isothermal parametrization $S(u, v)$:

- $F = S_u \cdot S_v = 0, \quad E = S_u^2 = G = S_v^2$
- Conformal
- Maps small squares from domain to small squares on the surface



$(\theta, \theta_0) = (arccos\frac{1}{\sqrt{5}}, 45°)$



Construction of I-nets

- I-nets from isothermal parametrizations

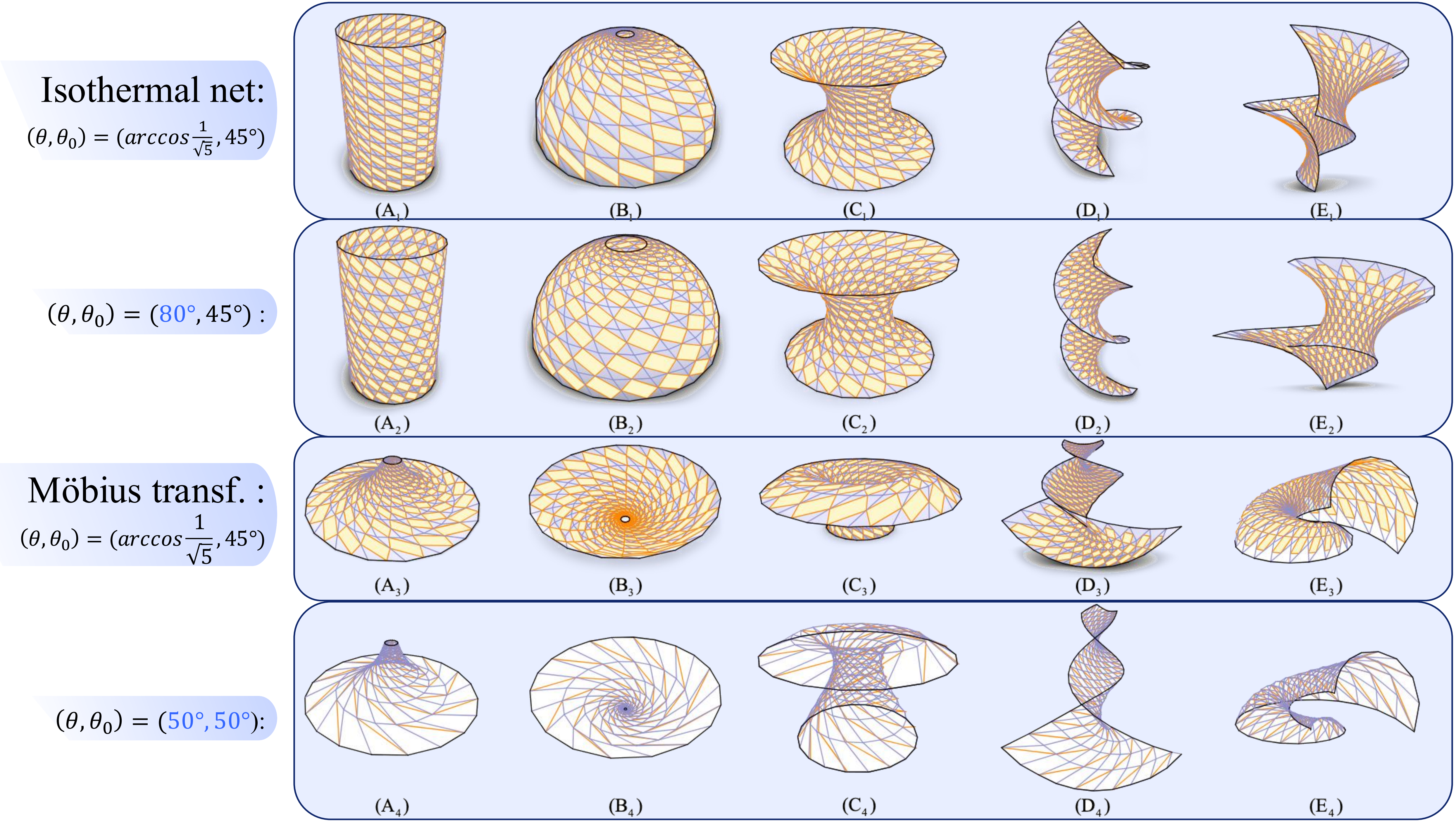
Surface of revolution: $S(u,v) = f(v)\cos u, f(v)\sin u, g(v))$ $\implies$ $F = 0$

$\boxed{E = f^2(v) = f'^2(v) + g'^2(v) = G}$

Isothermal parametrizations

- Cylinder: $f(v) = r, g(v) = r\,v$
- Sphere: $f(v) = r\,\mathrm{sech}v, g(v) = r\,\tanh v$
- Catenoid (C): $f(v) = r\cosh v, g(v) = r\,v$

- Helicoid (H): $S(u,v) = (r\,\sinh v\,cos u, r\,\sinh v\,sin\,u, ru)$
- Conjugate minimal surface: $(\cos t)\,C + (\sin t)\,H, \quad t \in \mathbb{R}$



Isothermal net:
$(\theta, \theta_0) = (arccos\frac{1}{\sqrt{5}}, 45°)$
(A$_1$)
(B$_1$)
(C$_1$)
(D$_1$)
(E$_1$)
$(\theta, \theta_0) = (80°, 45°)$:
(A$_2$)
(B$_2$)
(C$_2$)
(D$_2$)
(E$_2$)
Möbius transf. :
$(\theta, \theta_0) = (arccos\frac{1}{\sqrt{5}}, 45°)$
(A$_3$)
(B$_3$)
(C$_3$)
(D$_3$)
(E$_3$)
$(\theta, \theta_0) = (50°, 50°)$:
(A$_4$)
(B$_4$)
(C$_4$)
(D$_4$)
(E$_4$)

Computational Design

- Initialization of I-nets
- Optimization

$$E_+ = E_\square + \omega_1 E_* + \omega_2 E_{fair} + \omega_3 E_{clos} + \omega_4 E_{app} + \omega_5 E_{bdry}$$

$$\frac{||v_4 - v_2||}{||v_3 - v_1||} = \lambda, \quad \frac{||v_3 + v_4 - v_1 - v_2||}{||v_1 + v_4 - v_2 - v_3||} = \mu$$

Assign const. $\bar{\lambda}$, $\bar{\mu}$ or $\bar{\theta}$, $\bar{\theta_0}$

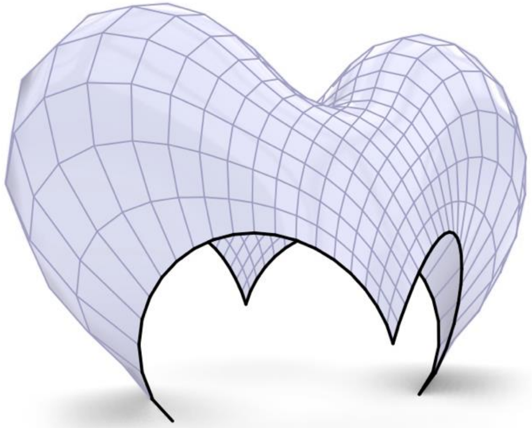
$$E_{fair} = \sum_{i \in polyline} (2\mathbf{v}_i - \mathbf{v}_{il} - \mathbf{v}_{ir})^2$$

$$E_{clos} = \sum_{i \in vertices} (\mathbf{v}_i - \bar{\mathbf{v}}_i)^2$$

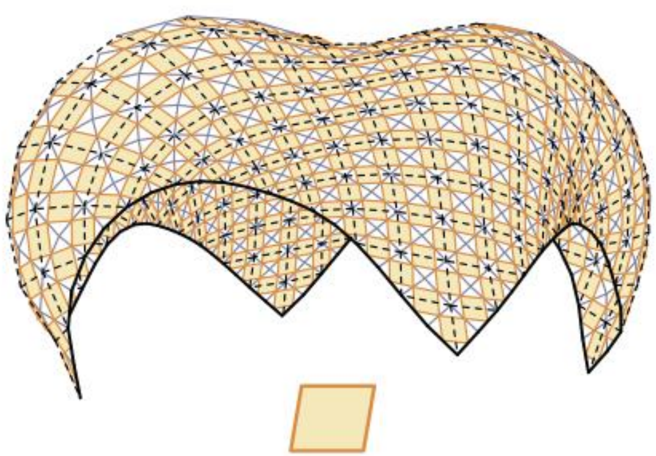
$$E_{app} = \sum_{i \in vertices} ((\mathbf{v}_i - \mathbf{p}_i) \cdot \mathbf{n}_i)^2$$

$$E_{bdry} = \sum_{i \in boundary} ((\mathbf{v}_i - \mathbf{p}_i) \times \mathbf{e}_{i1})^2$$

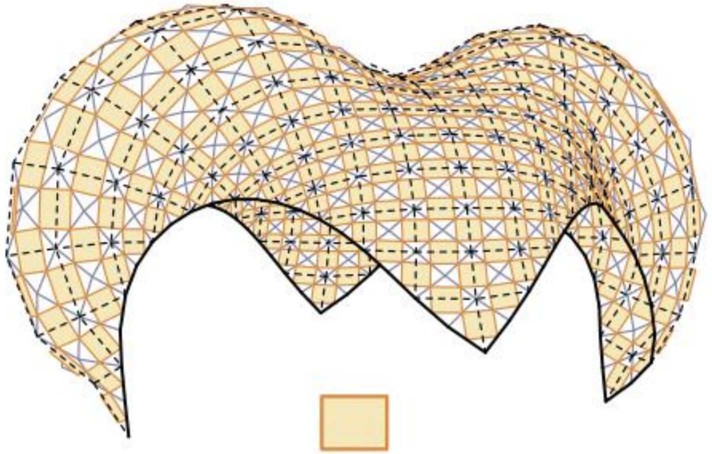
Results



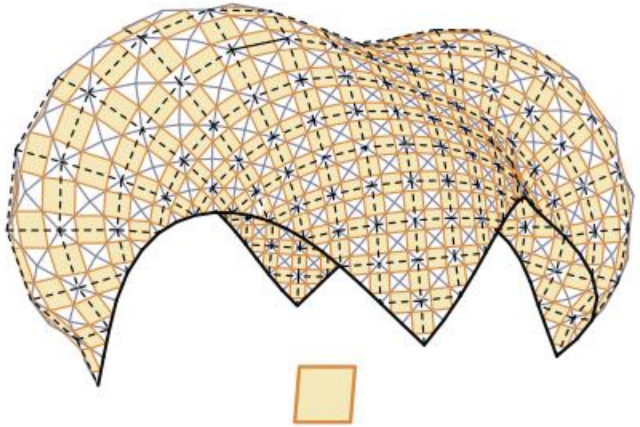
Initial mesh



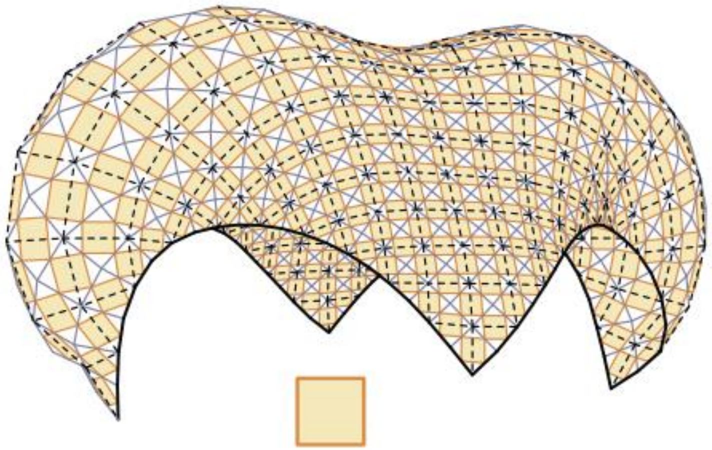
(A) $(\theta, \theta_0) = (80.42°, 84.30°)$



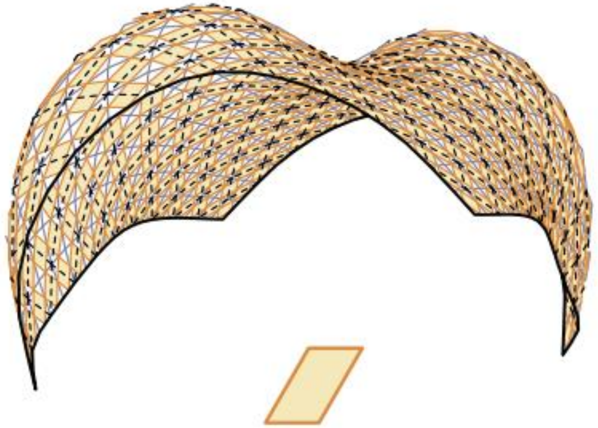
(B) $(\theta, \theta_0) = (90°, 77.92°)$



(C) $(\theta, \theta_0) = (85.27°, 90°)$

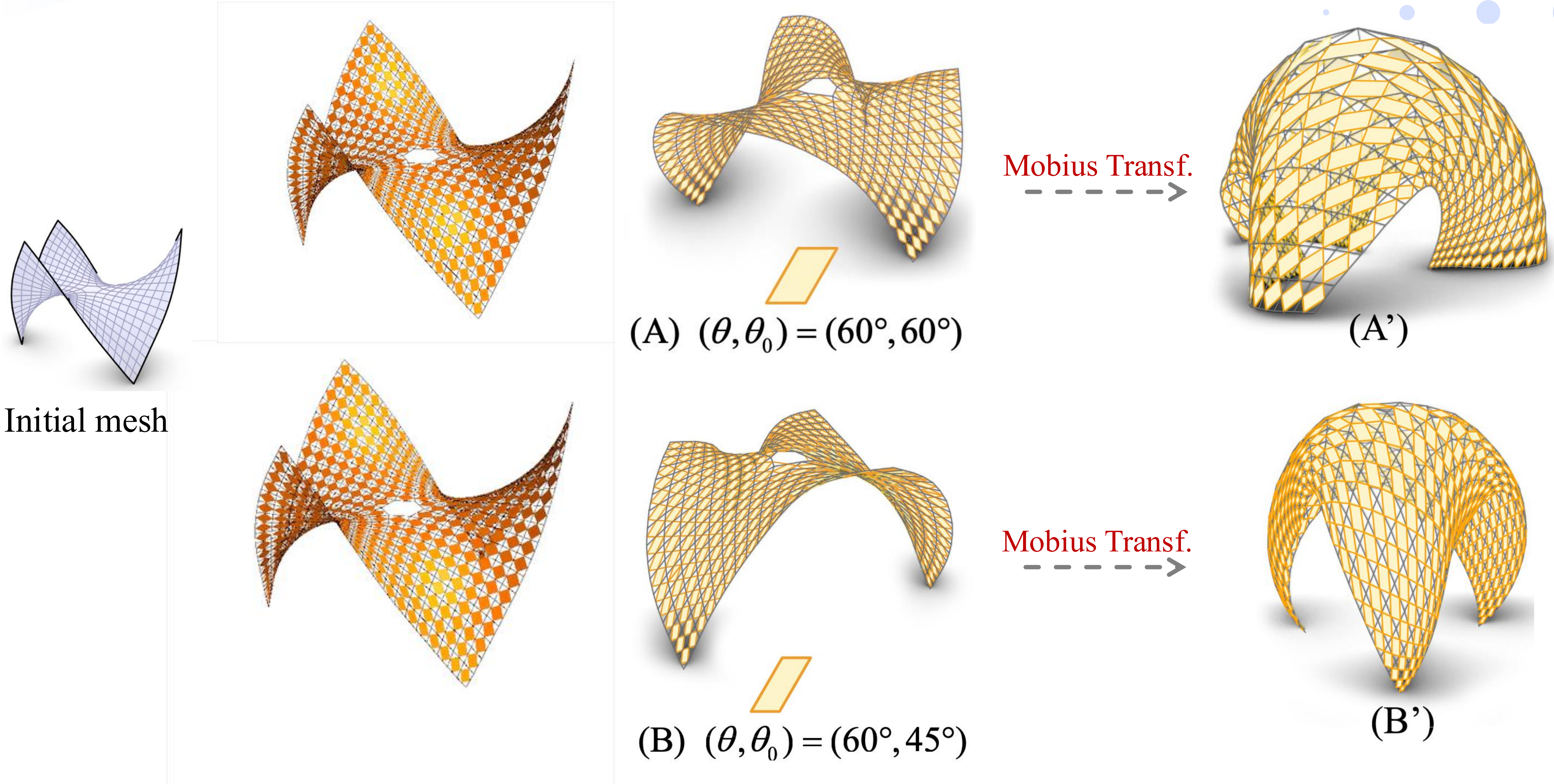


(D) $(\theta, \theta_0) = (90°, 90°)$



(E) $(\theta, \theta_0) = (60°, 60°)$

Results


Initial mesh
(A) $(\theta, \theta_0) = (60°, 60°)$
Mobius Transf.
(A')
(B) $(\theta, \theta_0) = (60°, 45°)$
Mobius Transf.
(B')

Conclusion

- A novel method for I-net and I-web using similar mid-edge subdivided parallelograms
- Expand the scope of geometric design possibilities and ensures compatibility with orthogonal nets
- Simplify the optimization process by constraining edge ratios instead of angles

Future work

- Integrate with other specialized nets, e.g. Weingarten surfaces
- Potential applications:
  - paneling design for freeform architectural surfaces
  - gridshell structures with uniform knots
  - repetitive pattern design
  - auxetic quads with similar elements
  - periodic structures in additive manufacturing
  - woven fabrics with regular patterns
  - ……

Thank you！

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GMP 2025

WashU

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